AHARONOV-BOHM EFFECT REVISITED

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. Aharonov-Bohm effect is a quantum mechanical phenomenon that attracted the attention of many physicists and mathematicians since the publication of the seminal paper of Aharonov and Bohm [1] in 1959.

We consider different types of Aharonov-Bohm effect such as magnetic AB effect, electric AB effect, combined electromagnetic AB effect, AB effect for the Schrödinger equations with Yang-Mills potentials, and the gravitational analog of AB effect.

We shall describe different approaches to prove the AB effect based on the inverse scattering problems, the inverse boundary value problems in the presence of obstacles, spectral asymptotics, and the direct proofs of the AB effect.

Keywords: Aharonov-Bohm effect, Schrödinger equation, gauge equivalence.

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Contents

1. Introduction

The Aharonov-Bohm effect was discovered by Aharonov-Bohm in the famous paper [1]. Consider the Schrödinger equation

$$-ih\frac{\partial u}{\partial t}+\frac{1}{2m}\sum_{j=1}^{n}\Big(-ih\frac{\partial}{\partial x_j}-\frac{e}{c}A_j(x)\Big)^2u+eV(x)u=0 \tag{1.1}$$

in the plane domain $(\mathbb{R}^2\setminus\Omega_1)\times(0,T)$, where

$$u\Big|_{\partial\Omega_1\times(0,T)}=0, \tag{1.2}$$

$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^2\setminus\Omega_1. \tag{1.3}$$

Here Ω_1 is a bounded domain in $\mathbb{R}^2$ called an obstacle. Equation (1.1) describes a nonrelativistic quantum electron in a classical electromagnetic field with time-independent magnetic potential $A(x)=(A_1(x),A_2(x))$ and electric potential $V(x)$.

Assume that the magnetic field $B(x)=\operatorname{curl}A(x)=0$ in $\mathbb{R}^2\setminus\Omega_1$, i.e. the magnetic field is shielded in Ω_1. Aharonov and Bohm have shown that despite the absence of the magnetic field in $\mathbb{R}^2\setminus\Omega_1$ the magnetic potential $A(x)$ has a physical impact.

They proposed the following physical experiment to demonstrate this fact:

Let a coherent beam of electrons splits into two parts and each part passes on the opposite sides of the obstacle Ω_1. Then both beams merge at the interferometer behind the obstacle Ω_1. The interference of these two beams allows to measure

$$\alpha = \frac{e}{hc}\int\limits_{\gamma} A(x)\cdot dx \quad (\text{modulo } \ 2\pi n) \tag{1.4}$$

where $n \in \mathbb{Z}$.

Here γ is a simple contour encircling Ω_1. The integral α is called the magnetic flux.

When two potentials $A^{(1)}(x)$ and $A^{(2)}(x)$ are such that $\operatorname{curl} A^{(1)} = \operatorname{curl} A^{(2)} = 0$ but

$$\frac{e}{hc}\int\limits_{\gamma}(A^{(1)}(x) - A^{(2)}(x))\cdot dx \neq 2\pi n, \quad n \in \mathbb{Z},$$

the magnetic potentials $A^{(1)}(x)$ and $A^{(2)}(x)$ make a different physical impact, since the measurements of the interferometer are different. This phenomenon is called the Aharonov-Bohm (AB) effect.

We shall present, following Wu and Yang [52], a more general formulation of the AB effect that can be applied to more general situations:

Let $G(\mathbb{R}^2 \setminus \Omega_1)$ be the group on $\mathbb{R}^2 \setminus \Omega_1$ consisting of all smooth complex-valued functions $g(x)$ such that $|g(x)| = 1$ in $\mathbb{R}^2 \setminus \Omega_1$ and $g(x) = e^{ip\theta(x)}\big(1 + O\big(\frac{1}{|x|}\big)\big)$ when $|x| \to \infty$. Here $p \in \mathbb{Z},\ O = (0,0) \in \Omega_1$ and $\theta(x)$ is the polar angle of x. The group $G(\mathbb{R}^2 \setminus \Omega_1)$ is called the gauge group. The map $u' = g^{-1}(x)u$ is called the gauge transformation. If $u(x,t)$ satisfies (1.1) and $u' = g^{-1}u$, then $u'(x,t)$ satisfies (1.1) with $A(x)$ replaced by

$$\frac{e}{c}A'(x) = \frac{e}{c}A(x) + ihg^{-1}\frac{\partial g}{\partial x}. \tag{1.5}$$

Two magnetic potentials related by (1.5) are called gauge equivalent. Any two magnetic potentials belonging to the same gauge equivalence class represent the same physical reality and cannot be distinguished in any physical experiment.

The Aharonov-Bohm effect is the statement that two magnetic potentials belonging to different gauge equivalent classes make a different physical impact.

The first mathematical proof of AB effect was given by Aharonov and Bohm in the original paper [1]. They found explicitly the scattering amplitude in the case when the obstacle Ω_1 is a point O and $A(x) = \frac{\alpha}{2\pi}\big(\frac{-x_2}{|x|}, \frac{x_1}{|x|}\big)$. They have shown that the scattering cross-section

is proportional to $\sin^2\frac{\alpha}{2}$ where α is the magnetic flux (1.4). Their result was extended by Ruijsenaars [43] to the case when Ω_1 is the disk $|x|\leq R$.

The further progress was done in the solution of the inverse scattering problem of defining the gauge equivalence class of magnetic potential knowing the scattering matrix (amplitude). We shall mention only the works when obstacles are present. Nicoleau [39], Weder [50], and Ballesteros and Weder [2] proved that the scattering matrix asymptotics for high energies in the dimensions 2 and 3 determines all integrals

$$\exp\Big(\frac{ie}{hc}\int\limits_{-\infty}^{\infty}A(x_0+t\omega)\cdot\omega dt\Big), \tag{1.6}$$

where $x=x_0+t\omega$ is any straight line that does not intersect the obstacle Ω_1. When the obstacle is convex they used the X-ray transform to determine the gauge equivalence class of the magnetic potential, in particular, to determine the magnetic field $B=\operatorname{curl}A$. Further, using the second term of high energy asymptotic of the scattering matrix and knowing $B(x)$, they were able to determine all integrals $\frac{e}{h}\int_{-\infty}^{\infty}V(x_0+t\omega)dt$. Thus, the X-rays transform allows to recover electric potential $V(x)$ outside a convex obstacle. See also a related work of Enss and Weder [9].

In [24] Eskin, Isozaki and O'Dell studied the inverse scattering problem for any number of obstacles, not necessary convex.

In [44], [45], [53] Yafaev, and Roux and Yafaev described the singularities of the scattering amplitude .

More on the inverse scattering problem see §2.4.

Another class of inverse problems are inverse boundary value problems.

Consider the stationary Schrödinger equation

$$\frac{1}{2m}\sum_{j=1}^{n}\Big(-ih\frac{\partial}{\partial x_j}-\frac{e}{c}A_j(x)\Big)^2w(x)+eV(x)w=k^2w(x) \tag{1.7}$$

in the domain $\Omega\setminus\Omega_1$, where

$$\begin{aligned} w\Big|_{\partial\Omega_1}&=0,\\ w\Big|_{\partial\Omega}&=f. \end{aligned} \tag{1.8}$$

The Dirichlet to Neumann (DN) operator $\Lambda(k)$ is the map of the Dirichlet data $f=w\big|_{\partial\Omega}$ to the Neumann data $\big(h\frac{\partial w}{\partial\nu}-i\,\frac{e}{c}A\cdot\nu w\big)\big|_{\partial\Omega}$ for

all smooth solutions of (1.6), (1.7), (1.8), i.e.

$$\Lambda(k)f = \Big(h\frac{\partial w}{\partial \nu} - i\,\frac{e}{c}A(x)\cdot \nu w\Big)\Big|_{\partial\Omega}, \tag{1.9}$$

where ν is the outward unit normal to Ω. Note that the group $G(\overline{\Omega}\backslash\Omega_1)$ consists of all smooth complex-valued $g(x)$ such that $|g(x)| = 1$.

The inverse boundary value problem consists of determining the gauge equivalence class of the magnetic potential and of determining the electric potential knowing the DN operator $\Lambda(k)$ on $\partial\Omega$.

One can consider also the case of several obstacles $\Omega' = \bigcup_{j=1}^r \Omega_j$ when $\overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$ where $j \neq k$, $\overline{\Omega}_j \subset \Omega, 1 \leq j \leq r$. Then $u\big|_{\partial\Omega'} = 0$ in (1.8) instead of $u\big|_{\partial\Omega_1} = 0$.

The inverse boundary value problems were studied intensively in many papers (see, for example, the monograph of Isakov [33] and references there). The case of domain with obstacles was considered in [11], [12], [13]. The strongest results were obtained by the reduction to the inverse boundary value problem for the hyperbolic equation $\big(\frac{h^2}{2m}\frac{\partial^2}{\partial t^2} + H\big)u = 0$, where H is the operator in the left hand side of (1.6), and using the Boundary Control (BC) method (see [5], [35], [36], [16], [17], [19]). This approach allows to solve the inverse boundary value problem in the case of any number of obstacles, not necessary convex. Moreover it is enough to know the DN operator only on an arbitrary open part of the boundary $\partial\Omega$. Also BC method allows to recover not only the gauge equivalent classes of magnetic potentials and the electric potentials, but also allows to recover the number and location of obstacles (see more details in §2.1).

Assuming that $\operatorname{curl} A = 0$ in $\Omega \setminus \Omega'$ we prove the AB effect in §2.1. Moreover, we prove that always when A and A' belong to distinct gauge equivalent classes they have a different physical impact.

In [10], [15], [18], [47] a more general class of Schrödinger equations with Yang-Mills potentials was considered, i.e. the equations of the form

$$\sum_{j=1}^{n}\Big(-i\frac{\partial}{\partial x_j}I_m - A_j(x)\Big)^2 u + V(x)u = k^2 u, \quad x \in \Omega\setminus\Omega', \tag{1.10}$$

where $u(x) = (u_1(x), ..., u_m(x)$ is m-vector, $A_j(x), 1 \leq j \leq n,\ V(x)$ are self-adjoint $m \times m$ matrices called the Yang-Mills potentials, I_m is the identity operator in $\mathbb{C}^m, \Omega' = \bigcup_{j=1}^r \Omega_j$. We assume that $u\big|_{\partial\Omega'} = 0$.

The Dirichlet-to-Neumann operator has the form

$$\Lambda(u\big|_{\partial\Omega}) = \Big(\frac{\partial}{\partial x} - iA(x)\Big)\cdot\nu(x)u(x)\big|_{\partial\Omega}, \tag{1.11}$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega, A = (A_1, ..., A_n)$.

The gauge group $G(\overline{\Omega}\setminus\Omega')$ consists of smooth unitary $m\times m$ matrices $g(x)$ on $\overline{\Omega}\setminus\Omega'$. Two Yang-Mills potentials $(A(x),V(x))$ and $(A'(x),V'(x))$ are gauge equivalent if there exists $g(x)\in G(\overline{\Omega}\setminus\Omega')$ such that

$$g^{-1}A(x)g(x)+i\frac{\partial g}{\partial x}g^{-1}=A'(x),\quad g^{-1}V(x)g=V'(x). \tag{1.12}$$

The Schrödinger equation with electromagnetic potentials is a particular case when $m=1$,

In [15] the BC method was applied to the equations of the form (1.10) and all results for the equation (1.6) were extended to the equations of the form (1.10) (see more in §2.2).

Note that the DN operator is not gauge invariant.

The gauge invariant boundary data on $\partial\Omega$ were found in [11], [18] using the probability density $|w(x)|^2$ and the probability current $S(w) = \Im\big(\frac{\partial w}{\partial x} - iA(x)w(x)\big)\overline{w}(x)$. It will be shown in §2.3 that

$$\begin{aligned} &|w(x)|^2\big|_\Gamma = f_1(x),\quad \frac{\partial}{\partial\nu}|w(x)|^2\big|_\Gamma = f_2(x),\\ &S(w)\big|_\Gamma = f_3(x) \end{aligned} \tag{1.13}$$

are gauge invariant boundary data that uniquely determine the gauge equivalence class of magnetic potential $A(x)$ and the electric potential $V(x)$. Here Γ is any open subset of $\partial\Omega$. Therefore if $A(x)$ and $A'(x)$ belong to distinct gauge equivalence classes then corresponding gauge invariant boundary data (1.13) will be different. This gives another proof of magnetic AB effect.

There is a close relationship between the inverse boundary value problems (IBVP) and the inverse scattering problem (ISP). We will assume that the magnetic field $B = \operatorname{curl} A$ and electric potential $V(x)$ have compact supports in the ball $B_R = \{|x| < R\}$. If also $\operatorname{supp} A(x) \subset B_R$ there is a general theorem (see §2.4) that the scattering amplitude $a(\theta,\omega,k)$ given for all $|\omega| = |\theta| = 1$ uniquely determine the DN operator $\Lambda(k)$ on $\{|x| = R\}$ and vice versa, i.e. the IBVP and ISP are equivalent.

When the flux $\alpha\neq 0$ the magnetic potential is not compactly supported and the relation between IBVP and ISP is more complicated (see §2.4 for details).

Another venue to test the AB effect is the spectrum of the magnetic Schrödinger operator. The first result in this direction belongs to Helffer [28] (see also [37]). He considered the magnetic Schrödinger operator of the form (1.7) in $\mathbb{R}^2\setminus\overline{\Omega}_1$ where $\Omega_1 = \{|x| < 1\}$, $\operatorname{curl} A = 0$ in $\mathbb{R}^2\setminus\overline{\Omega}_1$ and $V(x)\to+\infty$ when $|x|\to\infty$. He has shown that the

lowest Dirichlet eigenvalue depends on the cosine of the magnetic flux (1.4). This proves the AB effect. In [25] the Schrödinger equation (1.7) in $\Omega\setminus\overline{\Omega}_1$ was considered where $\Omega=\{x:|x|<R\}$, R is large, with the zero Dirichlet conditions on $\partial\Omega$ and $\partial\Omega_1$ and $\operatorname{curl}A=0$ in $\Omega\setminus\Omega_1$. It was proven that Dirichlet spectrum also depends on $\cos\alpha$, where α is the magnetic flux, thus proving the AB effect.

Note that the AB effect holds always when the domain is not simply-connected even if there are no obstacles. For example, in [25] the AB effect is demonstrated for the Schrödinger operator of the form (1.6) on the torus (see [25] and §2.5).

All methods to prove the AB effect described above are quite complicated.

A direct and simple proof of the AB effect was proposed in [21]. It essentially mimics the AB experiment (see §2.6 and Remark 3.1 in [21], see also papers of Ballesteros and Weder [3], [4] on the justification of AB experiment).

The AB effect holds also for $n\geq 3$ dimensions, for example, when the domain is $\mathbb{R}^3\setminus\Omega_1$, where $\partial\Omega_1=T^2$ is the two dimensional torus and the magnetic field is zero outside Ω_1 (see §2.6). Note that the most accurate AB type experience was done by Tonomura et al [T] for such domain.

It is important also to study the case of several obstacles $\Omega_1,...,\Omega_m$ in $\mathbb{R}^2$ where $\overline{\Omega}_j\cap\overline{\Omega}_k=\emptyset$ when $j\neq k$. Suppose we have the magnetic field shielded inside $\Omega_j, 1\leq j\leq m$, and $B=\operatorname{curl}A=0$ outside of all obstacles. Let $\alpha_j=\frac{e}{hc}\int_{\gamma_j}A\cdot dx$ be the fluxes corresponding to each obstacles Ω_j. Here γ_j is a simple contour encircling Ω_j only. Suppose that some $\alpha_j\neq 2\pi n, \forall n\in\mathbb{Z}$, but the total flux $\sum_{j=1}^m\alpha_j=0$ (modulo $2\pi n$). Suppose that the obstacles are close to each other and therefore we can not perform AB experiment separately for each Ω_j. From other side the treatment of $\bigcup_{j=1}^m\Omega_j$ as one obstacle does not reveal the AB effect since the total flux is zero modulo $2\pi n$. The AB effect in this case was proven in [13], [21] using broken rays solutions. We were able to recover all magnetic fluxes $\alpha_j, j=1,2,...,m$, up to a sign.

The magnetic AB effect is studied in the hundreds of papers (see the survey [42]). In the original paper [1] Aharonov and Bohm discuss also the electric AB effect. They consider the Schrödinger equation with time-dependent electric potential and zero magnetic potential

$$ih\frac{\partial u(x,t)}{\partial t}+\frac{h^2}{2m}\Delta u(x,t)-eV(x,t)u(x,t)=0. \tag{1.14}$$

In contrast with hundreds of papers on the magnetic AB effect there are only few papers dealing with the electric AB effect. In particular,

in [51] Weder studied the electric AB effect assuming that the electric potential depends on a large parameter.

Let domain $D \subset \mathbb{R}^n \times [0,T]$. Denote by D_{t_0} the intersection of D with the plane $t = t_0$ We assume that

$$u\big|_{\partial D_t} = 0 \text{ for } 0 < t < T \text{ and } u(x,0) = u_0(x) \text{ on } D_0.$$

We assume that the electric field $E = \frac{\partial V}{\partial x} = 0$ in D. If D_{t_0} is connected for all $t_0 \in (0,T)$ then $\frac{\partial V(x,t_0)}{\partial x} = 0$ implies that $V(x,t_0) = V(t_0)$ is independent of x in D.

Consider a gauge transformation

$$w(x,t) = \exp\Big(i\frac{e}{h}\int\limits_0^t V(t')dt'\Big)u(x,t),$$

where $u(x,t)$ is the solution of (1.14). Then $w(x,t)$ satisfies the Schrödinger equation

$$ih\frac{\partial w(x,t)}{\partial t} + \frac{h^2}{2m}\Delta u(x,t) = 0.$$

Note that $w\big|_{\partial D_t} = 0$ for $0 < t < T$ and $w(x,0) = u_0(x)$ on D_0.

Therefore the electric potential $V(x,t)$ is gauge equivalent to zero electric potential if $E = \frac{\partial V}{\partial x} = 0$ in D and D_t are connected for all $t \in (0,T)$. This explains why there was no neither experimental nor mathematical evidence of AB effect in the situation when the domain D has the form $D = \Omega \times (0,T)$ where Ω is a domain in $\mathbb{R}^n$. For the electric AB effect to take place one need to consider domains with moving boundaries, i.e. D_t changes with t and is connected for some t and is disconnected for other $t, t \in (0,T)$ (cf. [21] and §3).

In §4 we study the Schrödinger equation with time-dependent electric and magnetic potentials.

Let $\Omega_j(t), 1 \le j \le r$, be obstacles in $\mathbb{R}^n$. Let $\Omega_0 \supset \overline{\Omega}_j(t),\ \forall t \in [0,T], 1 \le j \le r,\ \Omega_0$ be a simply-connected bounded domain in $\mathbb{R}^n, \Omega'(t) = \bigcup_{j=1}^r \Omega_j(t), \Omega' = \bigcup_{0 \le t \le T} \Omega'(t)$.

Consider the Schrödinger equation

$$\Big(ih\frac{\partial u}{\partial t} - Hu\Big) = 0 \text{ in } (\Omega_0 \times (0,T)) \setminus \Omega', \tag{1.15}$$

where

$$H = \frac{1}{2m}\sum_{j=1}^n \Big(-ih\frac{\partial}{\partial x_j} - \frac{e}{c}A_j(x,t)\Big)^2 + eV(x,t),$$

$A(x,t) = (A_1, ..., A_n)$ and $V(x,t)$ are magnetic and electric potentials.

We assume that

$$u(x,0) = 0 \ \text{ in } \ \Omega_0 \setminus \Omega'(0), \tag{1.16}$$

$$u\big|_{\partial\Omega'(t)} = 0, \quad 0 \le t \le T, \quad u\big|_{\partial\Omega_0\times(0,T)} = f. \tag{1.17}$$

We first consider the inverse boundary value problem for (1.15), (1.16), (1.17). The gauge group $G((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$ consists of $g(x,t) \in C^\infty((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$ such that $|g(x,t)| = 1$. Since coefficients of (1.15) are time-dependent we can not reduce (1.15) to the hyperbolic equation and apply BC method as in §2.1. We use a more traditional approach (cf. [18]) consisting of two steps:

a) Construction of geometric optics type solution for the Schrödinger equation with time-depending coefficients that are concentrated in a small neighborhood of a ray or a broken ray. This part can be done under quite mild restrictions on the geometry of obstacles (cf. [15], [18]).

b) In the second step one needs to study the injectivity of the X-ray type transform in the domain with obstacles. The presence of obstacles makes the results quite restrictive.

If the geometric conditions on obstacles are satisfied one can prove (cf. [18] and §4.1) that if there are two Schrödinger equations $\Big(ih\frac{\partial u_k}{\partial t} - H_k u\Big) = 0, k = 1, 2,$ of the form (1.15) with initial and boundary conditions (1.16), (1.17) and if corresponding DN operators $\Lambda_k, k = 1, 2,$ are gauge equivalent on $\partial\Omega_0 \times (0,T)$ then electromagnetic potentials $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are also gauge equivalent.

Consider now the equation (1.15) in unbounded domain $(\mathbb{R}^n \times (0,T)) \setminus \Omega'$ with the initial condition

$$\begin{aligned} &u(x,0) = u_0(x) \ \text{ in } \ \mathbb{R}^n \setminus \Omega'(0), \\ &u\big|_{\partial\Omega'(t)} = 0, \quad 0 \le t \le T. \end{aligned}$$

We will assume that $u_0(x) = 0$ in $\Omega_0 \setminus \Omega'(0)$ as in (1.16).

In this case the gauge group $G((\mathbb{R}^n \times (0,T)) \setminus \Omega')$ consists of $|g(x,t)| = 1$ in $\mathbb{R}^n \times [0,T] \setminus \Omega'$ and we assume that $g(x,t)$ are independent of t in $(\mathbb{R}^n \setminus \Omega_0) \times [0,T]$. When $n \ge 3$ we also assume that $g(x) = \exp\big(\frac{i}{h}\varphi(x)\big)$ for $|x| > R$, where $\varphi(x)$ is real-valued, $\varphi(x) = O\big(\frac{1}{|x|}\big)$.

When $n = 2$ we assume that $g(x) = e^{ip\theta(x)}\big(1 + O\big(\frac{1}{|x|}\big)\big)$ for $|x| > R$.

Note that $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are gauge equivalent if

$$
\begin{aligned}
(1.18) \qquad & \frac{e}{c}A^{(2)}(x,t) = \frac{e}{c}A^{(1)}(x,t) + ihg^{-1}\frac{\partial g}{\partial x}, \\
& eV^{(2)}(x,t) = eV^{(1)}(x,t) - ihg^{-1}\frac{\partial g}{\partial t}.
\end{aligned}
$$

Since coefficients of the equation (1.15) are time-dependent, the scattering operator for H is not defined. We propose a new inverse problem in $(\mathbb{R}^n \times (0,T) \setminus \Omega'$ instead of the inverse scattering problem.

Let $u(x,t)$ be the solution of (1.15) in $(\mathbb{R}^n \times (0,T)) \setminus \Omega'$, and let $(A(x,t), V(x,t))$ be independent of t for $|x| > R$. We assume that

$$
(1.19) \qquad u(x,0) \ \text{ and } \ u(x,T)
$$

are known on $\mathbb{R}^n \setminus B_R$.

Then (see Theorem 4.3 in §4.2) these two times ($t = 0$ and $t = T$) data determine $u(x,t)$ in $((\mathbb{R}^n \setminus B_R) \times (0,T)$.

More precisely, the following result holds:

Let $ih\frac{\partial u_k}{\partial t} - H_k u_k = 0, k = 1,2$, be two equations of the form (1.15) in $(\mathbb{R}^n \times (0,T)) \setminus \Omega'$. Suppose corresponding electromagnetic potentials $(A^{(k)}, V^{(k)}), k = 1,2$, are independent of t for $|x| > R$ and gauge equivalent with some gauge $g_0(x)$.

Suppose the two times ($t = 0$ and $t = T$) data (1.19) of $u_1(x,t)$ and $u_2(x,t)$ are gauge equivalent, i.e.

$$
u_2(x,0) = g_0(x)u_1(x,0), \quad u_2(x,T) = g_0(x)u_1(x,T), \quad x \in \mathbb{R}^n \setminus B_R.
$$

Then

$$
(1.20) \qquad u_2(x,t) = g(x)u_1(x,t) \ \text{ in } \ (\mathbb{R}^n \setminus B_R) \times (0,T).
$$

The relation (1.20) implies that the DN operators Λ_1 and Λ_2 are gauge equivalent on $\partial B_R \times (0,T)$. Then assuming that the geometric conditions on obstacles formulated in Theorem 4.1 are satisfied, the electromagnetic potentials $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are gauge equivalent.

Note that as in the case of time-independent magnetic and electric potentials it is naturally to consider the gauge invariant boundary data as in §2.3.

We shall mention also the inverse boundary value problems for the time-dependent Yang-Mills potentials. The powerful BC method used in the case of time-independent Yang-Mills potentials can not be applied here. However, we can solve the inverse boundary problem using the method of Non-Abelian Radon transforms developed in [11], [14], [40]. This method does not work, unfortunately, if the obstacles are present.

Now we shall consider the AB effect for time-dependent electromagnetic potentials assuming that $B = \operatorname{curl} A(x,t) = 0$ and $E = -\frac{1}{c}\frac{\partial A(x,t)}{\partial t} - \frac{\partial V(x,t)}{\partial x} = 0$ in $(\Omega_0 \times (0,T)) \setminus \Omega'$.

Let $\alpha_\gamma = \int_\gamma A(x,t)\cdot dx - V(x,t)dt$ be electromagnetic flux where γ is a closed contour in $(\Omega_0 \times (0,T)) \setminus \Omega'$. It follows from $E = B = 0$ and the Stoke's theorem that α_γ depends only on the homotopy class of γ in $(\Omega_0 \times (0,T)) \setminus \Omega'$.

Let $\gamma_1, ..., \gamma_m$ be the basis of the homology group of $(\Omega_0 \times (0,T)) \setminus \Omega'$, i.e. any closed contour in $(\Omega_0 \times (0,T)) \setminus \Omega'$ is homotopic to a linear combination of $\gamma_1, ..., \gamma_m$ with integer coefficients.

Denote $\alpha_{\gamma_k} = \frac{e}{hc}\int\limits_{\gamma_k} Adx - Vdt, \quad 1 \leq k \leq m.$

Two electromagnetic potentials $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are gauge equivalent if and only if

$$\alpha_{\gamma_p}^{(1)} = \alpha_{\gamma_p}^{(2)} \ (\mathrm{mod}\ 2\pi n, n \in \mathbb{Z}), \quad 1 \leq p \leq m. \tag{1.21}$$

Here $\alpha_{\gamma_p}^{(k)} = \frac{e}{hc}\int\limits_{\gamma_p} A^{(k)}dx - V^{(k)}dt, \quad k = 1,2$. Therefore to demonstrate the electromagnetic AB effect we will need to check only a finite number of relations (1.21).

Thus we do not need to prove the injectivity of the X-ray transform to demonstrate the AB effect. Therefore we can relax the restriction on the geometry of obstacles imposed in Theorem 4.1. Moreover, we can consider a more general class of obstacles.

We shall consider a class of domains $D^{(1)}$ with obstacles that may move and may merge or split at some times t_k, $1 \leq k \leq l$ (see Fig. 4). The intersections of $D^{(1)}$ with $t = t_0$ are connected for each $t_0 \in (0,T)$. We denote by $D^{(2)}$ a more general class of domains obtained from $D^{(1)}$ by making holes in some obstacles (cf. §4.5). Now the intersection of $D^{(2)}$ with $t = t_0$ may be not connected for some $t_0 \in (0,T)$ and hence the combined AB effect takes place.

A simple examples of domains of the type $D^{(1)}$ is the following domain $D_0^{(1)}$: let $\Omega_0 = \{x_1^2 + x_2^2 < r^2\}, D_0^{(1)} \cap \{t = t_0\} = \Omega_0 \setminus \Omega_1(t_0)$, where $\Omega_1(t_0)$ is the obstacle moving with the speed v_1 along x_1-axis: $\Omega_1(t) = \{(x_1 - v_1 t)^2 + x_2^2 < r_1^2\}, r_1 \leq r$ and small, $0 \leq t \leq T$. We assume that $\Omega_1(T) \subset \Omega_0$.

Let $\omega = \{(x_1 - \frac{r_1}{2})^2 + (x_2 - \frac{r_1}{2})^2 < \frac{r_1^2}{16}\}$. The hole H in $D_0^{(1)}$ is the intersection of the cylinder $\omega \times (0,T)$ with $\bigcup_{0\leq t\leq 1}\Omega_1(t)$. Therefore the domain of class $D^{(2)}$ is $D_0^{(1)} \cup H$.

If $B = \operatorname{curl} A = 0$ and $E = -\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial V}{\partial x} = 0$ in $D_0^{(1)}$ then $\alpha_\gamma = \int_\gamma A(x,t)\cdot dx - V(x,t)dt$ is the same for any closed contour γ in $D^{(1)}$

encircling the obstacles. Any such γ is homotopic to a contour γ_0 in the plane $t = \text{const}$ encircling the obstacle, i.e. γ_0 is the basis of the homology group in $D_0^{(1)}$. In the case of $D_0^{(2)}$ there are two contours that form the basis for the homology group in $D_0^{(2)}$. One of them is γ_0 and the second is any closed contour γ_1 that is passing through the hole H and not shrinking to a point.

In $D_0^{(1)}$ the potentials $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are having a different physical impact if

$$(1.22)\quad \alpha = \frac{e}{hc}\int\limits_{\gamma_0}(A^{(1)}-A^{(2)})dx \text{ or } -\alpha \text{ are not equal to } 2\pi p,\ \forall p \in \mathbb{Z}.$$

In $D^{(2)}$ $(A^{(1)}, V^{(1)})$ and $(A^{(2)}, V^{(2)})$ are having a different physical impact if either (1.22) holds or $A^{(1)}$ and $A^{(2)}$ are gauge equivalent and

$$(1.23)\qquad \frac{e}{h}\int\limits_{\gamma_1}\frac{A^{(1)}-A^{(2)}}{c}\cdot dx - (V^{(1)}-V^{(2)})dt \neq 2\pi p,\ \forall p \in \mathbb{Z}.$$

These two examples are a particular case of general results in §4.4 and §4.5.

An important part of the proof of the AB efect is the construction of geometric optics type solution in $D^{(1)} = (\Omega_0 \times (0,T)) \setminus \Omega'$ similar to the solutions for the solving inverse boundary value problem (see §4.1).

These geometric optics type solutions are the solutions of (1.15) in $D^{(1)}$ only and have nonzero Dirichlet data on $\partial\Omega_0 \times (0,T)$. It is not clear what is their physical meaning. From the other side, the solutions of (1.15), (1.16), (1.17) in $(\mathbb{R}^n \times (0,T)) \setminus \Omega'$ describe the electron in the magnetic field shielded by obstacles Ω' and therefore are physically meaningful. It is proven in §4.5 (the density lemma 4.5) that any solution of (1.15) in $D^{(1)} = (\Omega_0 \times (0,T)) \setminus \Omega'$ can be approximated by the restrictions to $D^{(1)}$ of physically meaningful solutions of (1.15), (1.16), (1.17) in $(\mathbb{R}^n \times (0,T)) \setminus \Omega'$. This allows to complete the proof of electromagnetic AB effect in §4.4.

The AB type effect holds not only in quantum mechanics but also in other branches of physics (cf. [6], [7], [49]). We shall consider the gravitational analog of AB effect extending the results of Stashel [46].

First, we reformulate the magnetic AB effect in $\mathbb{R}^2 \setminus \Omega_1$ assuming, for the simplicity of notations, that $h = e = c = 1$. Suppose $B = \operatorname{curl} A = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = 0$ in $\mathbb{R}^2 \setminus \Omega_1$. If $\omega \subset \mathbb{R}^2 \setminus \Omega_1$ is a simply connected subdomain of $\mathbb{R}^2 \setminus \Omega_1$ then $\frac{\partial A_1}{\partial x_2} - \frac{\partial A_2}{\partial x_1} = 0$ in ω implies that there exists $\Psi(x_1, x_2)$ in ω such that $A_1 = \frac{\partial \Psi}{\partial x_1}, A_2 = \frac{\partial \Psi}{\partial x_2}$, i.e. $A(x)$ is the gradient of $\Psi(x_1, x_2)$. Making the gauge transformation $u' = e^{i\Psi}u$ we get the

Schrödinger equation with zero magnetic potential in ω, i.e. there is no AB effect in ω. The AB effect takes place when curl $A = 0$ but $A(x)$ is not a gradient in $\mathbb{R}^2 \setminus \Omega_1$ and has a flux $\alpha \neq 2\pi n,\ \forall n \in \mathbb{Z}$.

Similar situation arise for the wave equation corresponding to a pseudo-Riemannian metric $\sum_{j=0}^{n} g_{jk}(x)dx_j dx_k$ with Lorentz signature, where x_0 is the time variable, $x = (x_1, ..., x_n) \in \Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \Omega_j$. We assume that $g_{jk}(x)$ are independent of x_0, i.e. the metric is stationary.

Consider the group of transformations

$$\begin{aligned} x' &= \varphi(x) \\ x_0' &= x_0 + a(x), \end{aligned} \tag{1.24}$$

where $\varphi(x)$ is a diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega}' = \varphi(\overline{\Omega})$ and $a(x) \in C^\infty(\overline{\Omega})$. Two metrics $\sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k$ and $\sum_{j,k=0}^{n} g_{jk}'(x')dx_j' dx_k'$ are called isometric if

$$\sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k = \sum_{j,k=0}^{n} g_{jk}'(x')dx_j' dx_k', \tag{1.25}$$

where (x_0', x_0) and (x', x) are related by (1.24).

The group of isometries plays the role of the gauge group for the electromagnetic AB effect.

We shall prove (cf. Theorem 5.2) that if two metrics are locally isometric but globally not isometric, then they have a different physical impact.

We also extend a result of [46] that if a metric is locally static but not globally static, then this fact also has a physical impact. This is a gravitational analog of AB effect (cf. §5 and [22]).

2. Magnetic AB effect

In this section we consider the most well-known magnetic AB effect and we will review the different approaches to study it.

2.1. **Inverse boundary value problems for the Schrödinger equation with time-independent electromagnetic potentials.**

Let Ω_0 be a smooth bounded domain in $\mathbb{R}^n$, and let $\Omega_j, 1 \leq j \leq r$, be the smooth obstacles inside Ω_0, $\overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$ when $j \neq k$. Consider a stationary Schrödinger equation in $\Omega_0 \setminus \Omega'$, where $\Omega' = \bigcup_{j=1}^{r} \Omega_j$:

(2.1)
$$Hw \overset{def}{=} \frac{1}{2m_j} \sum_{j=1}^{n} \Big(-ih\frac{\partial}{\partial x_j} - \frac{e}{c}A_j(x)\Big)^2 w(x) + eV(x)w(x) = k^2 w(x),$$

$$w\big|_{\partial\Omega'} = 0, \tag{2.2}$$

$$(2.3)\qquad w\big|_{\partial\Omega_0} = f.$$

If k does not belong to a discrete set N of the Dirichlet eigenvalues then the Dirichlet-to-Neumann (DN) operator

$$(2.4)\qquad \Lambda(k)f = \Big(h\frac{\partial w}{\partial\nu} - i\frac{e}{c}A(x)\cdot\nu(x)w\Big)\Big|_{\partial\Omega_0}$$

is well-defined bounded operator from $H_{\frac{3}{2}}(\partial\Omega_0)$ to $H_{\frac{1}{2}}(\partial\Omega_0)$, where $H_s(\partial\Omega_0)$ is a Sobolev space of order s on $\partial\Omega_0$. Note that $\Lambda(k)$ is analytic in k on $\mathbb{C}\setminus N$. Thus the knowledge of $\Lambda(k)$ on any small interval $(k_0-\varepsilon, k_0+\varepsilon)$ determines $\Lambda(k)$ for all $k\in\mathbb{C}\setminus N$.

Let $\Gamma\in\partial\Omega_0$ be an open subset of $\partial\Omega_0$. We say that $\Lambda(k)$ is given on Γ if the restriction $\Lambda(k)f\big|_\Gamma$ is known for any f with support in $\overline{\Gamma}$.

Theorem 2.1. *Suppose two Schrödinger equations* $(H-k^2)w=0$ *and* $(H'-k^2)w'=0$ *are given in* $\Omega_0\setminus\bigcup_{j=1}^{r'}\Omega_j$ *and* $\Omega_0\setminus\bigcup_{j=1}^{r'}\Omega_j'$ *with electromagnetic potentials* $(A(x),V(x))$ *and* $(A'(x),V'(x))$*, respectively. Suppose the corresponding DN operators* $\Lambda(k)$ *and* $\Lambda'(k)$ *coincide on* Γ *for* $k\in(k_0-\varepsilon,k_0+\varepsilon)$*. Then* $A'(x)$ *and* $A(x)$ *are gauge equivalent with the gauge* $g(x)$ *equal to 1 on* Γ*,* $V'(x)=V(x), r'=r$ *and* $\Omega_j'=\Omega_j, 1\leq j\leq r$*.*

The proof of Theorem 2.1 is based on the reduction to the hyperbolic inverse boundary value problem and use of the powerful Boundary Control method for solving such problems (cf. Belishev [5], Kachalov-Kurylev-Lassas [35], Eskin [16], [17]):

Consider the initial-boundary value problem for the hyperbolic equation

$$(2.5)\qquad \frac{h^2}{2m}\frac{\partial^2 v}{\partial t^2} + Hv = 0,\quad x\in\Omega_0\setminus\bigcup_{j=1}^{r}\Omega_j,\ 0<t<+\infty,$$

with zero initial conditions

$$(2.6)\qquad v(x,0) = \frac{\partial v}{\partial t}(x,0) = 0,\quad x\in\Omega_0\setminus\Omega',$$

and boundary conditions

$$(2.7)\qquad v\big|_{\partial\Omega_j'\times(0,+\infty)} = 0,\ 1\leq j\leq r,\quad v\big|_{\partial\Omega_0\times(0,+\infty)} = \varphi(x',t),$$

where $\varphi(x',t)$ has a compact support on $\Gamma\times(0,+\infty)$.

Define the hyperbolic DN operator Λ_H as

$$(2.8)\qquad \Lambda_H\,\varphi = \Big(h\frac{\partial v}{\partial\nu} - i\frac{e}{c}A(x)\cdot\nu v\Big)\Big|_{\Gamma\times(0,+\infty)}.$$

The following result holds (see, for example, Theorem 1.1 in [16]):

Theorem 2.2. *Consider two hyperbolic equations* $\big(\frac{h^2}{2m}\frac{\partial^2}{\partial t^2}+H\big)v=0$, $\big(\frac{h^2}{2m}\frac{\partial^2}{\partial t^2}+H'\big)v'=0$ *in* $(\Omega_0\backslash\bigcup_{j=1}^r\Omega_j)\times(0,+\infty),\ (\Omega_0\backslash\bigcup_{j=1}^{r'}\Omega_j')\times(0,+\infty)$, *respectively, with zero initial conditions (2.6) and with the boundary conditions*

$$v\big|_{\partial\Omega_0\times(0,+\infty)}=\varphi,\quad v\big|_{\Omega_j\times(0,+\infty)}=0,\quad 1\leq j\leq r, \tag{2.9}$$

and

$$v'\big|_{\partial\Omega_0\times(0,+\infty)}=\varphi',\quad v'\big|_{\Omega_j'\times(0,+\infty)}=0,\quad 1\leq j\leq r', \tag{2.10}$$

respectively, where $supp\,\varphi\subset\overline{\Gamma}$, $supp\,\varphi'\subset\overline{\Gamma}$.

If the hyperbolic DN operator Λ_h *and* Λ_h' *are equal on* $\Gamma\times(0,+\infty)$, *i.e.* $\Lambda_h\varphi=\Lambda_h'\varphi$ *on* $\Gamma\times(0,+\infty)$ *for any* φ *with the support in* $\Gamma\times[0,+\infty)$, *then* $A(x)$ *and* $A'(x)$ *are gauge equivalent with the gauge* $g=1$ *on* Γ, $V(x)=V'(x), r=r'$ *and* $\Omega_j=\Omega_j',\ 1\leq j\leq r$.

Remark 2.1 Theorem 1.1 in [16] states that there exists a diffeomorfism $x'=\psi(x)$ of $\overline{\Omega}_0\setminus\bigcup_{j=1}^r\Omega_r$ onto $\overline{\Omega}_0\setminus\bigcup_{j=1}^{r'}\Omega_j',\ \psi(x)=x$ on Γ and $\sum_{j=1}^n(dx_j)^2=\sum_{j=1}^n(dx_j')^2$, where $x'=\psi(x)$. This implies that $\psi=I$ and therefore $r=r'$ and $\Omega_j=\Omega_j',\ j=1,...,r$.

To prove Theorem 2.1 we take the Fourier transform in t. Then the equation (2.5) becomes the equation (2.1) and the hyperbolic DN operator Λ_h on $\Gamma\times(0,+\infty)$ becomes the DN operator $\Lambda(k)$ on Γ.

We shall use Theorem 2.1 to prove the magnetic AB effect.

Suppose $(H_k-\lambda^2)w_k=0,\ k=1,2$, are two Schrödinger equations of the form (2.1) with electromagnetic potentials $A^{(k)}(x),V^{(k)}(x)),k=1,2$, respectively. Suppose $V^{(1)}=V^{(2)}=V$ and $\operatorname{curl}A^{(k)}=0$ in $\Omega_0\setminus\Omega',\ k=1,2$. Fix a point $x_0\in\Gamma$ and let ω be a simply-connected neighborhood of x_0. Let $\omega_+=\Omega_0\cap\omega$ and suppose $\Gamma=\partial\Omega\cap\omega$. Since $\operatorname{curl}A^{(k)}=0$ in $\overline{\omega}_+$ and ω_+ is simply-connected, there exists a smooth $\Psi_k(x)$ in $\overline{\omega}_+$ such that $A^{(k)}=\frac{\partial\Psi_k}{\partial x}$ in $\overline{\omega}_+,\ k=1,2$. Let $\tilde{\Psi}_k$ be a smooth extension of $\Psi_k(x)$ to $\overline{\Omega}_0\setminus\Omega'$ and let $g_k(x)=e^{-\frac{ie}{hc}\tilde{\Psi}_k}$. Then making the gauge transformation with the gauge $g_k(x)$ we transform H_k to $\hat{H}_k,\ k=1,2$, where $\hat{H}_k$ has electromagnetic potentials $(\hat{A}^{(k)},\hat{V}^{(k)})$ such that $\hat{V}^{(k)}(x)=V(x),\ \hat{A}^{(k)}=0$ in $\overline{\omega}_+,\ k=1,2$. Therefore $\hat{H}_1=\hat{H}_2$ in ω_+. Now we shall prove the magnetic AB effect.

Theorem 2.3. *Magnetic potentials* $\hat{A}^{(1)}$ *and* $\hat{A}^{(2)}$ *(and consequently* $A^{(1)}$ *and* $A^{(2)}$*) are not gauge equivalent if and only if there exists* $f_0\in C_0^\infty(\Gamma)$ *such that*

$$\hat{\Lambda}^{(1)}f_0\big|_\Gamma\neq\hat{\Lambda}^{(2)}f_0\big|_\Gamma, \tag{2.11}$$

where $\hat{\Lambda}^{(k)}$ *are DN operators corresponding to* $\hat{H}_k$.

It follows from (2.11) that when $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ are not gauge equivalent they have different physical impact, i.e. AB effect holds.

Proof of Theorem 2.3: Suppose $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ are not gauge equivalent. If (2.11) does not hold, i.e. $\hat{\Lambda}^{(1)}f = \Lambda^{(2)}f$ on Γ for all $f \in C_0^\infty(\Gamma)$ then by Theorem 2.1 $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ must be gauge equivalent, and this is a contradiction. Vice versa, suppose (2.11) holds but $\hat{A}^{(1)}$ and $\hat{A}^{(2)}$ are gauge equivalent with some gauge $g(x)$. Since $\hat{A}^{(1)} = \hat{A}^{(2)} = 0$ in $\overline{\omega}_+$ we get from (1.5) that $g(x) = e^{i\alpha}$ in $\overline{\omega}_+$, where α is an arbitrary real constant. Let $\hat{u}_k$ be the solutions of $(\hat{H}_k - \lambda^2)\hat{u}_k = 0, \hat{u}_k\big|_{\partial\Omega_0} = f,\ f \in C_0^\infty(\Gamma),\ u_k\big|_{\partial\Omega'} = 0$. Since $g = e^{i\alpha}$ on $\overline{\omega}_+$ and $\hat{u}_1\big|_{\partial\Omega_0} = \hat{u}_2\big|_{\partial\Omega_0} = f$, we get that $\hat{u}_1 = \hat{u}_2$ in ω_+. Therefore $\hat{\Lambda}^{(1)}f\big|_\Gamma = \hat{\Lambda}^{(2)}f\big|_\Gamma$ for all $f \in C_0^\infty(\Gamma)$, and this contradicts (2.11). $\square$

2.2. **Inverse boundary value problems for the Schrödinger equation with time-independent Yang-Mills potentials.**
Consider the Schrödinger equation with Yang-Mills potentials (cf. (1.10)):
(2.12)
$$\sum_{j=1}^{n}\Big(-i\frac{\partial}{\partial x_j} - A_j(x)\Big)^2 w(x) + V(x)w(x) = k^2 w(x), \quad x \in \Omega_0 \setminus \bigcup_{j=1}^{r}\Omega_j,$$
where Yang-Mills potentials $A_j(x), 1 \le j \le n,\ V(x)$ are $m \times m$ self-adjoint matrices. The gauge group $G(\overline{\Omega}_0 \setminus \bigcup_{j=1}^r \Omega_j)$ consists of all unitary $m \times m$ matrices and two Yang-Mills potentials $(A(x), V(x))$ and $(A'(x), V'(x))$ are gauge equivalent if there exists $g(x) \in G(\overline{\Omega}_0 \setminus \bigcup_{j=1}^r \Omega_j)$ such that (1.12) holds.

We assume that

$$w\big|_{\partial\Omega_j} = 0, \quad 1 \le j \le r, \qquad w\big|_{\partial\Omega_0} = f, \quad \operatorname{supp} f \subset \overline{\Gamma}. \tag{2.13}$$

The following theorem generalizes Theorem 2.1 for the case of Yang-Mills potentials.

Theorem 2.4. *Let* $(H - k^2 I_n)w = 0$ *and* $(H' - k^2 I_m)w' = 0$ *be two Schrödinger equations corresponding to Yang-Mills potentials* $(A(x), V(x))$ *and* $(A'(x), V'(x))$, *respectively.* $(H - k^2)w = 0$ *is considered in* $\Omega_0 \setminus \bigcup_{j=1}^r \Omega_j$ *with boundary conditions* $w\big|_{\partial\Omega_j} = 0, 1 \le j \le r,\ w\big|_{\partial\Omega_0} = f$ *and* $(H' - k^2)w' = 0$ *is considered in* $\Omega_0 \setminus \bigcup_{j+1}^{r'} \Omega_j'$ *with boundary conditions* $w'\big|_{\partial\Omega_0} = f',\ w'\big|_{\partial\Omega_j'} = 0,\ 1 \le j \le r'$. *Let* Γ *be an open subdomain of* $\partial\Omega_0$. *Suppose that DN operators* $\Lambda(k)f = \big(\frac{\partial w}{\partial\nu} - iA\cdot\nu w\big)\big|_\Gamma$ *and* $\Lambda'(k)f = \big(\frac{\partial w'}{\partial v} - iA'\cdot\nu w'\big)\big|_\Gamma$ *coincide on* Γ,

i.e. $\Lambda'(k)f\big|_\Gamma = \Lambda(k)f\big|_\Gamma$ *for any* f *with the support in* $\overline{\Gamma}$ *and all* $k \in (k_0 - \varepsilon, k_0 + \varepsilon)$. *Then* $(A'(x), V'(x))$ *are gauge equivalent to* $(A(x), V(x))$, $r' = r$ *and* $\Omega_j' = \Omega_j, 1 \le j \le r$.

It was shown in [15] that the proof of Boundary Control method, given in [16], [17], extends to the hyperbolic equation with Yang-Mills potentials. Therefore analog of Theorem 2.2 holds and this implies that Theorem 2.4 is also true.

Remark 2.1 In the Theorem 2.4 we assumed that the DN operators $\Lambda(k)$ and $\Lambda'(k)$ are equal on the interval $(k_0-\varepsilon, k_0+\varepsilon)$ and therefore are equal for all k because they are analytic in k.

When $n \ge 3$, $\Gamma = \partial\Omega_0$ and there is no obstacles, a stronger results was proven in [10] that the Yang-Mills potentials $(A(x), V(x))$ and $(A'(x), V'(x))$ are gauge equivalent if $\Lambda'(k_0) = \Lambda(k_0)$ for a fixed k_0. The proof requires a different idea (see [10] and some simplifications of the proof in [26]).

2.3. **Gauge invariant boundary data.**
Let $u(x)$ be the solution of (2.1), (2.2). There are two basic gauge invariant quantities in quantum mechanics: the probability density $|u(x)|^2$ and the probability current

$$S(u) = \Im\Big(h\frac{\partial u}{\partial x} - i\frac{e}{c}Au\Big)\overline{u}. \tag{2.14}$$

The probability density is obviously gauge invariant since $|u'|^2 = |g^{-1}(x)u|^2 = |u|^2$ for any $g \in G(\overline{\Omega}_0 \setminus \bigcup_{j=1}^r \Omega_j)$. For the probability current we have

$$S(u') = \Im\Big(h\frac{\partial}{\partial x}(g^{-1}u) - \frac{ie}{c}A'g^{-1}u\Big)g\overline{u}$$
$$=\Im\Big(h\frac{\partial u}{\partial x}g^{-1} - hg^{-2}\frac{\partial g}{\partial x}u - i\Big(\frac{e}{c}A + ihg^{-1}\frac{\partial g}{\partial x}\Big)g^{-1}u\Big)g\overline{u}.$$

We used above that $\overline{g} = g^{-1}$ and that $\frac{e}{c}A' = \frac{e}{c}A + ihg^{-1}\frac{\partial g}{\partial x}$ (cf. (1.5)). Therefore $S'(u') = \Im\big(h\frac{\partial u}{\partial x} - i\frac{e}{c}Au\big)\overline{u} = S(u)$.

Using the probability density and the probability current we define gauge invariant data on $\partial\Omega_0$ for any solution $u(x)$ of (2.1), (2.2):

$$|u(x)|^2\big|_{\partial\Omega_0} = f_1(x'), \quad \frac{\partial}{\partial\nu}|u(x)|^2\big|_{\partial\Omega_0} = f_2(x'), \quad S(u)\big|_{\partial\Omega_0} = f_3(x'). \tag{2.15}$$

Lemma 2.5. *Consider all* $u(x)$ *and* $u'(x)$ *such that* $(H-k^2)u = 0$ *in* $\Omega_0 \setminus \Omega'$, $u\big|_{\partial\Omega'} = 0$, *and* $(H'-k^2)u' = 0$ *in* $\Omega_0 \setminus \Omega'$, $u'\big|_{\partial\Omega'} = 0$, *respectively. Let* Λ, Λ' *be the corresponding DN operators. Suppose*

that the set (f_1, f_2, f_3) of all gauge invariant boundary data for $u(x)$ and $u'(x)$ is the same.

Then there exists $g_0(x) \in G(\overline{\Omega}_0 \setminus \bigcup_{j=1}^r \Omega_j)$ such that

$$g_0\big|_{\partial\Omega_0}\Lambda'\big((g_0^{-1}u)\big|_{\partial\Omega_0}\big) = \Lambda\big(u\big|_{\partial\Omega_0}\big) \tag{2.16}$$

for all $u(x)$ such that $(H-k^2)u = 0,\ u\big|_{\partial\Omega'} = 0$.

Proof: Consider smooth $u_0(x), u_0'(x)$ having the same boundary data (2.15) and such that $|u_0(x)| = |u_0'(x)| > 0$ on $\partial\Omega_0$. Let $g_0(x) = \frac{u_0(x)}{u_0'(x)}$ near $\partial\Omega_0$. Extend $g_0(x)$ to the whole domain $\Omega_0 \setminus \Omega'$ keeping $|g_0(x)| = 1$. We have on $\partial\Omega_0$

$$\begin{aligned} S(u_0') &= \Im\big(hg_0^{-1}\frac{\partial u_0}{\partial x} - hg_0^{-2}\frac{\partial g_0}{\partial x}u_0 - i\frac{e}{c}A'(x)g_0^{-1}u_0\big)g_0\overline{u}_0 \\ &= S(u_0) + \Im\big(-hg_0^{-1}\frac{\partial g_0}{\partial x} + i\frac{e}{c}(A(x) - A'(x)\big)|u_0|^2. \end{aligned}$$

Since $S(u_0') = S(u_0)$ and since $g_0^{-1}\frac{\partial g_0}{\partial x}$ is imaginary, we get

$$-hg_0^{-1}\frac{\partial g_0}{\partial x} = i\frac{e}{c}(A'(x) - A(x)) \quad \text{when} \quad x \in \partial\Omega_0. \tag{2.17}$$

Analogously, let $u(x), u'(x)$ be any solutions of $(H-k^2)u = 0,\ (H'-k^2)u' = 0$ having the same boundary data and such that $|u(x)| = |u'(x)| > 0$.

Denote $g(x) = \frac{u(x)}{u'(x)}$. Then $u' = g^{-1}u$ on $\partial\Omega$ and analogously to (2.17) we get $hg^{-1}\frac{\partial g}{\partial x} = i\frac{e}{c}(A(x) - A'(x))$. Therefore $g^{-1}\frac{\partial g}{\partial x} = g_0^{-1}\frac{\partial g_0}{\partial x}$. Thus $\frac{\partial}{\partial x}\big(\frac{g}{g_0}\big) = 0$ on $\partial\Omega_0$. Hence $g = e^{i\alpha}g_0$ where α is a constant.

We have

$$\begin{aligned} (\Lambda' u'\big|_{\partial\Omega_0})\overline{u}'\big|_{\partial\Omega} &= \Big(h\frac{\partial u'}{\partial\nu} - i\frac{e}{c}A'\cdot\nu u'\Big)\overline{u}'\Big|_{\partial\Omega_0} \\ &= h\frac{1}{2}\frac{\partial|u'|^2}{\partial\nu} + i\Im\Big(h\frac{\partial u'}{\partial\nu} - i\frac{e}{c}A'\cdot\nu u'\Big)\overline{u}'\Big|_{\partial\Omega_0} \\ &= \Big(h\frac{1}{2}\frac{\partial}{\partial\nu}|u'|^2 + iS'(u')\nu\Big)\Big|_{\partial\Omega_0}. \end{aligned}$$

We used above that $\Re\frac{\partial u'}{\partial x}\overline{u}' = \frac{1}{2}\frac{\partial|u'|^2}{\partial x}$.

Analogously,

$$\Lambda\big(u\big|_{\partial\Omega_0}\big)\overline{u}\big|_{\partial\Omega_0} = \Big(h\frac{1}{2}\frac{\partial|u|^2}{\partial\nu} + iS(u)\cdot\nu\Big)\Big|_{\partial\Omega_0}.$$

Since $S'(u') = S(u)$ we get

$$\Lambda'\big(u'\big|_{\partial\Omega_0}\big)\overline{u}'\big|_{\partial\Omega_0} = \Lambda\big(u\big|_{\partial\Omega_0}\big)\overline{u}\big|_{\partial\Omega_0}$$

for all u, u' having the same boundary data and $|u| = |u'| > 0$ on $\partial\Omega_0$. Since $u' = e^{-i\alpha} g_0^{-1} u$ we get, cancelling $\overline{u}$ and $e^{-i\alpha}$ that $g_0 \Lambda'\big(g_0^{-1} u\big|_{\partial\Omega_0}\big) = \Lambda\big(u\big|_{\partial\Omega_0}\big)$. Since $u,\ |u| > 0$ on $\partial\Omega_0$ are dense in $L_2(\partial\Omega_0)$ we have that (2.16) holds for all $u(x)$, i.e. Lemma 2.5 is proven.

We shall call DN operators Λ and Λ' satisfying (2.16) gauge equivalent with the gauge g_0.

If potentials $A(x)$ and $A'(x)$ are gauge equivalent with gauge g then DN operator Λ and Λ' are also gauge equivalent with the same gauge. Indeed, on $\partial\Omega_0$ we have

$$\begin{aligned}
&\Lambda'\big(u'\big|_{\partial\Omega_0}\big) = \Big(h\frac{\partial u'}{\partial x} - i\frac{e}{c}A'u'\Big)\cdot\nu\Big|_{\partial\Omega_0} \\
&= \Big(hg^{-1}\frac{\partial u}{\partial x} - hg^{-2}u\frac{\partial g}{\partial x}\Big)\nu\Big|_{\partial\Omega_0} - i\Big(\frac{e}{c}A(x) + ih\frac{\partial g}{\partial x}g^{-1}\Big)\nu g^{-1}u\Big|_{\partial\Omega_0} \\
&= \Big(h\frac{\partial u}{\partial x} - i\frac{e}{c}Au\Big)\cdot\nu g^{-1}\Big|_{\partial\Omega_0} = g^{-1}\big|_{\partial\Omega_0}\Lambda\big(u\big|_{\partial\Omega_0}\big),
\end{aligned}$$

i.e. Λ' and Λ are gauge equivalent. The converse statement is also true.

Lemma 2.6. *Suppose DN operators Λ and Λ' are gauge equivalent with gauge g_0, i.e. (2.16) holds. Then magnetic potentials $A(x)$ and $A'(x)$ are also gauge equivalent with some gauge $g(x)$ and $V(x) = V'(x)$.*

Proof: Consider Schrödinger equations $(H-k^2)u = 0,\ (H'-k^2)u' = 0$ corresponding to potentials $(A, V),\ (A', V')$, respectively. Let Λ, Λ' be the corresponding DN operators. In $(H-k^2)u = 0$ make the gauge transformation $u_0 = g_0^{-1}u$. Then we obtain the Schrödinger operator $(H_0 - k^2)u_0 = 0$ where (A_0, V_0) are gauge equivalent to (A, V). Note that the DN operator corresponding to H_0 has the form $\Lambda_0\big(u_0\big|_{\partial\Omega_0}\big) = g_0^{-1}\big|_{\partial\Omega_0}\Lambda(g_0u_0)\big|_{\partial\Omega_0}$. It follows from (2.16) that $\Lambda' = \Lambda_0$. Therefore the DN operator for $(H' - k^2)u' = 0$ and $(H_0 - k^2)u_0 = 0$ are the same. By Theorem 2.1 the potentials (A', V') and (A_0, V_0) are gauge equivalent with some gauge g_1.

Therefore the potentials (A, V) and (A', V') are also gauge equivalent with gauge g_1g_0.

Combining Lemmas 2.5 and 2.6 we get that if gauge invariant data for (A, V) and (A', V') are equal as in Lemma 2.5 then (A, V) and (A', V') are gauge equivalent.

Remark 2.2. Lemmas 2.5 and 2.6 hold when we replace $\partial\Omega_0$ by any open subset $\Gamma \subset \partial\Omega_0$. Thus we have the following theorem:

Theorem 2.7. *Let $u(x), u'(x)$ and Λ, Λ' be the same as in Lemma 2.5. If the set of the gauge invariant boundary data on Γ for $u(x)$ and $u'(x)$*

is the same, then the magnetic potentials $A(x)$ and $A'(x)$ are gauge equivalent and

$$V(x) = V'(x).$$

The converse statement is obvious: if $(A(x), V(x)$ are gauge equivalent to $(A'(x), V'(x))$ then the set of boundary data (2.15) on Γ is the same for $u(x)$ and $u'(x)$ because the boundary data (2.15) are gauge invariant.

Theorem 2.7 has the corollary that gives another proof of the magnetic AB effect:

Corollary 2.1. *Suppose $\operatorname{curl} A = 0$, $\operatorname{curl} A' = 0$ and $V(x) = V'(x)$. If $A(x)$ and $A'(x)$ are not gauge equivalent then the sets of boundary data (2.15) are different for $u(x)$ and $u'(x)$. This implies that $A(x)$ and $A'(x)$ have a different physical impact, i.e. the AB effect holds.*

2.4. **Inverse scattering problems.**
We consider the Schrödnger equation (2.1) in $\mathbb{R}^n \setminus \bigcup_{j=1}^r \Omega_j$ assuming that

$$u\big|_{\partial\Omega_j} = 0, \quad 1 \leq j \leq r.$$

In problems related to AB effect the magnetic field $B = \operatorname{curl} A$ is shielded inside the obstacles Ω_j, $1 \leq j \leq r$, and therefore has a compact support in $\mathbb{R}^n$. The electric potential $V(x)$ plays no role in magnetic AB effect and could be taken even equal to zero. We assume that $V(x)$ also has a compact support. The magnetic potential $A(x)$ may have or may have not a compact support if $B(x)$ has a compact support.

Lemma 2.8. *Let $B(x)$ has a compact support, $\operatorname{supp} B(x) \subset B_R$. If $n \geq 3$ or if $n = 2$ and*

$$\iint\limits_{|x|<R} B(x) dx_1 dx_2 = 0, \tag{2.18}$$

then there exists a magnetic potential $A(x)$ with compact support such that $\operatorname{curl} A = B$ in $\mathbb{R}^n$ and $\operatorname{supp} A(x) \subset B_R$.

Proof: Consider first the case $n = 2$ and $\iint_{|x|<R} B(x) dx_1 dx_2 = 0$. Let $\tilde{B}(\xi) = \int_{\mathbb{R}^2} B(x) e^{-ix\cdot\xi} dx$ be the Fourier transform of $B(x)$. Since $\operatorname{supp} B(x) \subset B_R$, $\tilde{B}(\xi_1, \xi_2)$ is an entire function of $(\xi_1, \xi_2) \in \mathbb{C} \times \mathbb{C}$ and $|\tilde{B}(\xi)| \leq Ce^{R|\Im\xi|}$, where $\Im\xi = (\Im\xi_1, \Im\xi_2)$. Since $\iint_{|x|<R} B(x) dx_1 dx_2 = 0$ we have $\tilde{B}(0,0) = 0$. Applying the mean value theorem we have

$$\tilde{B}(\xi_1, \xi_2) = \xi_1 \tilde{B}_1(\xi) + \xi_2 \tilde{B}_2(\xi), \tag{2.19}$$

where

$$\tilde{B}_j(\xi_1,\xi_2) = \int_0^1 \frac{\partial \tilde{B}}{\partial \xi_j}(t\xi_1, t\xi_2)dt, \quad j = 1,2. \tag{2.20}$$

Obviously, $B_j(\xi_1,\xi_2)$ are also entire functions of (ξ_1,ξ_2) and

$$|\tilde{B}_j(\xi)| \leq Ce^{R|\Im\xi|}, \; j = 1,2. \tag{2.21}$$

By the Paley-Wiener theorem the inverse Fourier transform $B_j(x) = F^{-1}\tilde{B}_j(\cdot)$ is also contained in B_R. We have $\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = B(x)$. Making the Fourier transform and using (2.19) we can take $\tilde{A}_1(\xi) = i\tilde{B}_2(\xi)$, $\tilde{A}_2(\xi) = -i\tilde{B}_1(\xi)$. Therefore $\operatorname{supp} A_j \subset B_R$ and $\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = B(x)$.

Now consider the case $n \geq 3$. The equation $\operatorname{curl} A = B$ has the following form after performing the Fourier transform

$$\xi_2\tilde{A}_3(\xi) - \xi_3\tilde{A}_2(\xi) = -i\tilde{B}_1, \quad -\xi_1\tilde{A}_3(\xi) + \xi_3\tilde{A}_1(\xi) = -i\tilde{B}_2, \tag{2.22}$$

$$\xi_1\tilde{A}_2(\xi) - \xi_2\tilde{A}_1(\xi) = -i\tilde{B}_3. \tag{2.23}$$

Note that $\operatorname{div} B = 0$, i.e. $\xi_1\tilde{B}_1(\xi)+\xi_2\tilde{B}_2(\xi)+\xi_3\tilde{B}_3(\xi) = 0$. In particular, we have $\tilde{B}_3(0,0,\xi_3) = 0$. Therefore, as in (2.19), we have $\tilde{B}_3(\xi) = \xi_1\tilde{B}_{31}(\xi) + \xi_2\tilde{B}_{32}(\xi) = 0$ and we choose $\tilde{A}_1(\xi) = i\tilde{B}_{32}(\xi)$, $\tilde{A}_2(\xi) = -i\tilde{B}_{31}(\xi)$. Substituting in (2.22) we get

$$\xi_2\tilde{A}_3(\xi) = -i\tilde{B}_1 - i\xi_3\tilde{B}_{31}, \tag{2.24}$$

$$\xi_1\tilde{A}_3(\xi) = +i\tilde{B}_2 + i\xi_3\tilde{B}_{32}, \tag{2.25}$$

Note that $\xi_1(-i\tilde{B}_1 - i\xi_3\tilde{B}_{31}) = \xi_2(i\tilde{B}_2 + i\xi_3 B_{32})$ since $\xi_1\tilde{B}_{31} + \xi_2\tilde{B}_2 + \xi_3(\xi_1\tilde{B}_3 + \xi_2\tilde{B}_{32}) = 0$.

Therefore

$$\tilde{A}_3 = \frac{-i\tilde{B}_1 - i\xi_3\tilde{B}_{31}}{\xi_2} = \frac{i\tilde{B}_2(\xi) + i\xi_3\tilde{B}_{32}}{\xi_1}. \tag{2.26}$$

It follows from (2.24), (2.25) that $\tilde{A}_3(\xi)$ is analytic when $\xi_2 \neq 0$ or $\xi_1 \neq 0$. Therefore by the theorem of removable singularity for analytic functions of several variables $\tilde{A}_3(\xi)$ is an entire analytic function. Since estimates of the form (2.19) hold, $A_3(x) = F^{-1}\tilde{A}_3(\xi)$ has the support in B_R.

Therefore we proved the existence of the magnetic potential with compact support such that $\operatorname{curl} A = B$.

Remark 2.3. A more careful analysis allows to conclude that if $\operatorname{supp} B \subset \Omega_0$, where Ω_0 is a convex domain, then $\operatorname{supp} A \subset \Omega_0$.

Lemma 2.9. *Let $n = 2$, $\operatorname{supp} B(x) \subset \Omega_0$, where Ω_0 is convex, $(0,0) \in \Omega_0$, $\iint\limits_{\Omega_0} B(x)dx_1dx_2 = \alpha_0 \neq 0$ Then there exists a magnetic potential $A(x)$ in $\mathbb{R}^2$ such that $\operatorname{curl} A = B$ and $A(x) = A_0(x)$ in $\mathbb{R}^2 \setminus \Omega_0$, where*

$$A_0(x) = \frac{\alpha_0}{2\pi}\frac{(-x_2, x_1)}{|x|^2}. \tag{2.27}$$

The potential (2.27) is called the AB potential (cf. [1]).

Proof: Note that $\operatorname{curl} A_0 = \alpha_0\delta(x)$ in $\mathbb{R}^2$. Let $A'(x)$ be a magnetic potential such that $\operatorname{curl} A' = B(x) - \alpha_0\delta(x)$ in $\mathbb{R}^2$. Since $\iint_{\Omega_0}(B(x) - \alpha_0\delta(x))dx = 0$, by Lemma 2.8 we can choose $A'(x)$ such that $\operatorname{supp} A' \subset \Omega_0$. Consider $A(x) = A_0(x) + A'(x)$. Then $\operatorname{curl} A = B$ in $\mathbb{R}^2$, $A = A_0(x)$ for $x \in \mathbb{R}^2 \setminus \Omega_0$. $\square$

Consider now the inverse scattering problem for the case when $A(x)$ and $V(x)$ have compact supports that are contained in $\{|x| < R - \varepsilon\}$. We assume also that all $\overline{\Omega}_j \subset \{|x| < R - \varepsilon\}$, $1 \leq j \leq r$.

A solution $w(x, k\omega)$ of the form

$$w(x, k\omega) = e^{ik\omega\cdot x} + \frac{a(\theta, \omega, k)e^{ik|x|}}{|x|^{\frac{n-1}{2}}} + O\Big(\frac{1}{|x|^{\frac{n+1}{2}}}\Big)$$

is called a distorted plane wave. Here $|\omega| = 1$, $\theta = \frac{x}{|x|}$, $a(\theta, \omega, k)$ is called the scattering amplitude. The existence of distorted plane wave is well-known (see, for example, [30] or [20]). For the case of magnetic potentials in domains with obstacles see [12], [41].

We consider the inverse scattering problem of determining of the gauge equivalence class of $A(x)$ and of $V(x)$ knowing the scattering amplitude $a(\theta, \omega, k)$ for fixed k and all $\theta \in S^{n-1}$, $\omega \in S^{n-1}$.

Consider simultaneously the inverse boundary value problem in the domain $B_R \setminus \bigcup_{j=1}^n \Omega_j$, where $B_R = \{|x| < R\}$. We assume that the Dirichlet problem in $B_R \setminus \bigcup_{j=1}^n \Omega_j$ has a unique solution. Then DN operator is well defined.

Theorem 2.10. *Consider two equations $(H - k^2)u = 0$, $(H' - k^2)u' = 0$ in $\mathbb{R}^n \setminus \bigcup_{j=1}^r \Omega_j$. Let $a(\theta, \omega, k)$ and $a'(\theta, \omega, k)$ be corresponding scattering amplitudes and let $\Lambda(k)$ and $\Lambda'(k)$ be the corresponding DN operators on $\partial B_R = \{|x| = R\}$. If $a(\theta, \omega, k) = a'(\theta, \omega, k)$ for fixed k and for all $(\theta, \omega) \in S^{n-1} \times S^{n-1}$, then $\Lambda(k) = \Lambda'(k)$ for the same k. Vice versa, if $\Lambda(k) = \Lambda'(k)$, then $a(\theta, \omega, k) = a'(\theta, \omega, k)$ for all $(\theta, \omega) \in S^{n-1} \times S^{-1}$.*

Proof: Assume $a(\theta, \omega, k) = a'(\theta, \omega, k)$. Let $w(x, k\omega)$ and $w'(x, k\omega)$ be corresponding distorted plane waves. Since $a = a'$ we have $w(x, k\omega) -$

$w'(x,k\omega)=O\big(\frac{1}{|x|^{\frac{n+1}{2}}}\big),\ |x|>R$. By the Rellich's lemma (see, for example, Lemma 35.2 in [20]) we get

$$w(x,k\omega)-w'(x,k\omega)=0 \quad \text{for} \quad |x|\geq R.$$

Differentiating in x we have

$$\frac{\partial}{\partial\nu}(w(x,k\omega)-\omega'(x,k\omega))\big|_{\partial B_R}=0,$$

where $\frac{\partial}{\partial\nu}$ is the unit normal to ∂B_R. Therefore we got that $\Lambda(k)w=\Lambda'(k)w'$ on ∂B_R for any distorted plane wave. It is known (see, for example, [20]) that the restrictions of the distorted plane waves on ∂B_R are dense in $L_2(\partial B_R)$. Therefore taking the closure we get that $\Lambda(k)f=\Lambda'(k)f$ for all $f\in L_2(\partial B_R)$ (cf. [20]).

The converse statement is also true:

If $\Lambda(k)=\Lambda'(k)$ on ∂B_R then $a(\theta,\omega,k)=a'(\theta,\omega,k)$ for all $(\theta,\omega)\in S^{n-1}\times S^{n-1}$. We shall omit the proof (cf. [20], [33]). Therefore combining the Theorem 2.10 with the Theorem 2.1 for $\Gamma=\partial B_R$ we get that if $a(\theta,\omega,k)=a'(\theta,\omega,k)$ for all $(\theta,\omega)\in S^{n-1}\times S^{n-1}$ then $A(x)$ and $A'(x)$ are gauge equivalent with the gauge $g(x)=1$ for $|x|\geq R,\ V(x)=V'(x),\ r=r',\ \Omega_j'=\Omega_j$ for $1\leq j\leq r$. Note that in the case when $\operatorname{supp} A(x)\subset B_R$ the gauge group in $\mathbb{R}^n\setminus\bigcup_{j=1}^r\Omega_j$ consists of $|g(x)|=1,\ g(x)=1$ for $|x|\geq R$. $\square$

Now consider the inverse scattering problem in the case $n=2$ and magnetic flux $\alpha\neq 0$. We consider magnetic potentials of the form (cf. [24]) $A(x)=A_0(x)+A_1(x)$, where $A_0(x)$ has the form (2.27) and $A_1(x)=O\big(\frac{1}{|x|^{1+\varepsilon}}\big),\varepsilon>0$. Note that $\operatorname{curl} A=B=0$ for $|x|>R$.

We can choose inside the gauge equivalence class the magnetic potential equal to $A_0(x)=\frac{\alpha(-x_2,x_1)}{2\pi|x|^2}$ for $|x|>R$ Since $A_0(x)=O\big(\frac{1}{|x|}\big)$ the scattering amplitude is a distribution and it has the form (cf.[1], [43], [44], [45])

$$a(\theta,\omega,k)=a_0(\theta-\omega)+a_1(\theta,\omega,k), \tag{2.28}$$

where

$$a_0(\theta)=\cos\frac{\alpha}{2}\delta(\theta)+\frac{i\sin\frac{\alpha}{2}}{\pi}\ p.v.\frac{e^{i\left[\frac{\alpha}{2\pi}\right]\theta}}{1-e^{i\theta}},\quad |a_1(\theta,\omega,k)|\leq C|\theta-\omega|^{-\varepsilon},\quad 0\leq\varepsilon<1. \tag{2.29}$$

Here $[\alpha]$ is the smallest integer larger or equal to α.

The following analog of Theorem 2.10 holds (cf [24]):

Theorem 2.11. *Let $(H-k^2)u=0$, $(H'-k')u'=0$ be two Schrödinger operators in $\mathbb{R}^2\setminus\bigcup_{j=1}^r\Omega_j$. Suppose $A_0=\frac{\alpha_0(-x_2,x_1)}{2\pi|x|^2}$, $A_0'=\frac{\alpha_0'(-x_2,x_1)}{2\pi|x|}$ for $|x|>R$. If $a(\theta,\omega,k)=a'(\theta,\omega,k)$ and if $\alpha_0=\alpha_0'$ then $A(x)$ and $A'(x)$ are gauge equivalent.*

Note that in Theorem 2.11 we require not only that $a=a'$ but also that the magnetic fluxes α_0 and α_0' are equal.

It was shown in [24] that if there is only one convex obstacle, $\Omega_1=\Omega_1'$, then $a=a'$ and $\alpha_0\neq 2\pi n$, $\forall n\in\mathbb{Z}$, implies that $\alpha_0'=\alpha_0$.

A similar result holds for the inverse boundary value problem: If $\Lambda(k)=\Lambda'(k)$ on ∂B_R then $A(x)$ and $A'(x)$ are gauge equivalent in $B_R\setminus\bigcup_{j=1}^r\Omega_j$ and $\alpha_0=\alpha_0'$. Indeed, by Theorem 2.1 $A(x)$ and $A'(x)$ are gauge equivalent with the gauge $g(x)$ such that $g(x)=1$ on ∂B_R. Thus $\int_{\partial B_R} g^{-1}\frac{\partial g}{\partial x}\cdot dx=0$ and therefore $\alpha_0=\frac{e}{hc}\int_{\partial B_R}A(x)\cdot dx$ is equal to $\alpha_0'=\frac{e}{hc}\int_{\partial B_R}A'(x)\cdot dx$.

Note that when $\alpha_0\neq 0$ the gauge group has the form

$$|g(x)|=1,\ x\in\mathbb{R}^2\setminus\bigcup_{j=1}^r\Omega_j,\quad g(x)=e^{ip\theta(x)}\Big(1+O\Big(\frac{1}{|x|}\Big)\Big),$$

where $p\in\mathbb{Z}$.

When we make a gauge transformation, the scattering amplitude changes

$$a'(\theta,\omega,k)=e^{-ip\theta}a(\theta,\omega,k)e^{-ip(\theta+\pi)}. \tag{2.30}$$

Consider the gauge equivalence class of scattering amplitude for the operator $H-k^2$. It was shown in [24] that when Ω_1 is a single convex obstacle and $\alpha\neq 2\pi n$, $\forall n\in\mathbb{Z}$, then there is one-to-one correspondence between gauge equivalence classes of magnetic potentials and gauge equivalence classes of scattering amplitudes.

2.5. **Aharonov-Bohm effect and the spectrum of the Schrödiger operator.**

In this subsection we shall show that $\cos\alpha$, where α is a magnetic flux, is determined by the spectrum. Therefore if $\cos\alpha_1\neq\cos\alpha_2$ for two Schrödinger operators then their spectra are different.

Let Ω be a convex obstacle in $\mathbb{R}^2$ containing the origin. Let $B_R=\{|x|<R\}$, where R is large. Consider the Schrödinger equation $(H-\lambda)u=0$ in the annulus domain $B_R\setminus\Omega$ with zero Dirichlet boundary conditions $u\big|_{\partial B_R}=0$, $u\big|_{\partial\Omega}=0$. Let $\lambda_1\leq\lambda_2\leq\lambda_3\leq ...$ be the Dirichlet

spectrum and let $E(x,y,t)$ be the hyperbolic Green function , i.e.

$$\begin{aligned}
&\Big(\frac{\partial^2}{\partial t^2}+H\Big)E(x,y,t)=0 \ \text{ for } \ t>0,\\
&E(x,y,t)\Big|_{\partial\Omega\times(0,+\infty)}=0, \ \ E(x,y,t)\Big|_{\partial B_R\times(0,+\infty)}=0,\\
&E(x,y,0)=\delta(x-y), \quad \frac{\partial E(x,y,0)}{\partial t}=0.
\end{aligned}$$

The following wave trace formula holds (cf. [8])

$$\operatorname{Tr}(t)\overset{def}{=}\sum_{j=1}^{\infty}\cos\sqrt{\lambda_j}t=\int\limits_{B_R\setminus\Omega}E(x,x,t)dx. \tag{2.31}$$

It was proven in [8], [27] that the singularities of the wave trace occur at the time $t=T$, where T is equal to the length of periodic null-geodesics. In our geometry the periodic null-geodesics are equilateral N-gones inscribed in the circle $|x|=R$, in particular, equilateral triangles with the side $R\sqrt{3}$. It was proven in [25] that at $t=3R\sqrt{3}$ the singularity of $\operatorname{Tr}(t)$ has the form

$$-2^{-\frac{5}{2}}3^{\frac{1}{4}}R^{\frac{3}{2}}\cos\alpha(t-3R\sqrt{3})_+^{-\frac{3}{2}}+O\big((t-3R\sqrt{3})^{-\frac{1}{2}}\big). \tag{2.32}$$

Here $\alpha=\int_\gamma A(x)\cdot dx$ is the magnetic flux, γ is any simple closed contour between $\partial\Omega$ and ∂B_R (α is independent of γ since we assume that $\operatorname{curl}A=0$ in $B_R\setminus\Omega$), $(t-3R\sqrt{3})_+^{-\frac{3}{2}}$ is a homogeneous of order $-\frac{3}{2}$ distribution equal to zero when $t-3R\sqrt{3}<0$. Similar formula holds (cf [25]) when the triangle is replaced by N-gone. Therefore the spectrum depends on the magnetic flux.

Aharonov-Bohm effect holds when the underlying manifold is not simply-connected even when there are no obstacles.

Consider the Schrödinger operator on the torus (cf. [25]). Let $L=\{m_1e_1+m_2e_2,\ m_1,m_2\in\mathbb{Z}\}$ be a lattice in $\mathbb{R}^2$ and let L^* be the dual lattice consisting of $\delta\in\mathbb{R}^2$ such that $\delta\cdot d\in\mathbb{Z}$ for all $d\in L$. Consider the Schrödinger operator

$$H=\Big(-i\frac{\partial}{\partial x}-A(x)\Big)^2+V(x) \ \text{ on the torus } \ \mathbb{T}^2=\mathbb{R}^2/L. \tag{2.33}$$

The potentials $A(x)$ and $V(x)$ are periodic, i.e. $A(x+d)=A(x)$, $V(x+d)=V(x)$ for all $x\in R^2$ and $d\in L$ and therefore there are defined on $\mathbb{T}^2=\mathbb{R}^2/L$. We assume that the magnetic field $B=\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}=0$.

Let γ_1, γ_2 be the basis of the homology group of $\mathbb{T}^2$. Denote

$$\alpha_j = \int\limits_{\gamma_j} A(x)\cdot dx, \quad j=1,2. \tag{2.34}$$

The gauge group $G(\mathbb{T}^2)$ consists of $g(x)\in C^\infty(\mathbb{T}^2)$. such that $|g(x)| = 1$. Any such $g(x)$ has the form $g(x) = e^{i\delta\cdot x + i\varphi(x)}$ where $\varphi(x)\in C^\infty(\mathbb{T}^2)$ and $\delta \in L^*$.

Two magnetic potentials $A(x)$ and $A'(x)$ are gauge equivalent if $A' = A + ig^{-1}(x)\frac{\partial g}{\partial x}$.

Theorem 2.12. *Let H and H' be two Schrödinger operators on $\mathbb{T}^2$ with electromagnetic potentials $(A(x), V(x))$ and $(A'(x), V'(x))$. Suppose $curl\, A = curl\, A' = 0$. Suppose that the spectrum of H and H' are the same. Then*

$$\cos\alpha_j = \cos\alpha_j', \quad j=1,2, \tag{2.35}$$

$$\textit{where} \quad \alpha_j = \int\limits_{\gamma_j} A(x)\cdot dx, \quad \alpha_j' = \int\limits_{\gamma_j} A'(x)\cdot dx, \quad j=1,2.$$

This demonstrates the AB effect on torus since the magnetic fluxes make a physical impact.

2.6. Direct proof of magnetic AB effect.

Consider the nonstationary Schrödinger equation

$$-ih\frac{\partial u}{\partial t} + \frac{1}{2m}\sum_{j=1}^n \left(-ih\frac{\partial}{\partial x_j} - \frac{e}{c}A_j(x)\right)^2 u + eV(x)u = 0, \tag{2.36}$$

in $(\mathbb{R}^n\setminus\Omega')\times(0,T)$ where $n\geq 2,\ \Omega' = \bigcup_{j=1}^r \Omega_j$,

$$u(x,0) = u_0(x), \tag{2.37}$$

$$u\Big|_{\partial\Omega_j\times(0,T)} = 0, \quad 1\leq j\leq r. \tag{2.38}$$

At first we shall study the case of one obstacles in $\mathbb{R}^3$. Suppose Ω_1 is a toroid and $\mathrm{curl}\, A = 0$ in $\mathbb{R}^n\setminus\Omega_1$. It was shown in §2.4 that we can choose $A(x)$ having a compact support. Let $x^{(0)}\notin\overline{\Omega}_1$ and $\theta\in S^2$ be a unit vector. Suppose $\theta_{\perp 1}, \theta_{\perp 2}$ are two unit vectors such that $\theta, \theta_{\perp 1}, \theta_{\perp 2}$ is an orthonormal basis in $\mathbb{R}^3$. Let $\chi_0(s)\in C_0^\infty(\mathbb{R}^1), \chi_0(s) = 1$ for $|s| < \frac{1}{2},\ \chi_0(s) = 0$ for $|s| > 1,\ \chi_0(s) = \chi_0(-s)$. It was proven in [21]

that there exists a solution of (2.36) of the form

(2.39)
$$u(x,t,\theta) = e^{-i\frac{mk^2t}{2h}+i\frac{mk}{h}x\cdot\theta}\chi_0\Big(\frac{(x-x^{(0)})\cdot\theta_{\perp 1}}{\delta_1}\Big)\chi_0\Big(\frac{(x-x^{(0)})\cdot\theta_{\perp 2}}{\delta_1}\Big)$$
$$\cdot \exp\Big(i\frac{e}{hc}\int_0^\infty \theta\cdot A(x-s'\theta)ds'\Big) + O(\varepsilon),$$

where $t \in (0,T)$, $T = O\big(\frac{1}{k^{\delta_1}}\big)$, k is large , δ_1 is small, $\varepsilon > 0$ can be chosen arbitrary small if k is large enough.

The support of $u(x,t,\theta)$ modulo $O(\varepsilon)$ is contained in a small neighborhood of the line $x = x^{(0)} + s\theta$.

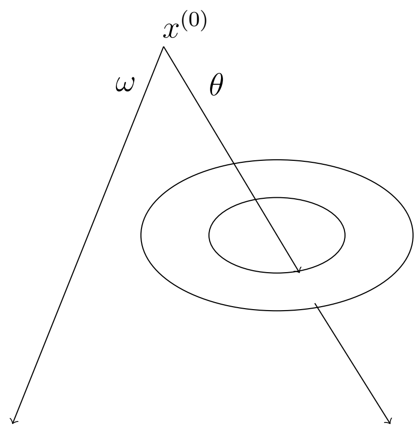

Fig. 1. Two rays $x = x^{(0)} + s\theta$, $x = x^{(0)} + s'\omega$, $0 \le s < +\infty$, $0 \le s' < +\infty$, intersect at point $x^{(0)}$. Only the ray $x = x^{(0)} + s\theta$ is passing through the hole of the toroid Ω_1.

We take two solutions $u(x,t,\theta)$ and $v(x,t,\omega)$ of the form (2.39) corresponding to the directions θ and ω, respectively (cf. Fig.1). Let U_0 be a ball of radius ε_0 centered at $x^{(0)}$. We have for $x \in U_0$

$$|u(x,t,\theta) - v(x,t,\omega)|^2 = \big|1 - e^{\frac{imk}{h}x(\theta-\omega)+i(I_1-I_2)}\big|^2 + O(\varepsilon), \tag{2.40}$$

where

$$I_1 = \frac{e}{hc}\int_0^\infty \theta\cdot A(x-s\theta)ds, \quad I_2 = \frac{e}{hc}\int_0^\infty \omega\cdot A(x-s\omega)ds. \tag{2.41}$$

Since $A(x)$ has a compact support and $\operatorname{curl} A = 0$ we have that $I_1 - I_2 = \alpha$, where $\alpha = \frac{e}{he}\int_\gamma A(x)\cdot dx$ is the magnetic flux, γ is a closed curve

passing through the hole and not shrinking to a point. Therefore

$$(2.42)\quad |u(x,t,\theta)-v(x,t,\omega)|^2 = 4\sin^2\frac{1}{2}\Big(\frac{mk}{h}x\cdot(\theta-\omega)+\alpha\Big)+O(\varepsilon).$$

Choose k_n large and such that $\frac{mk_n}{h}x^{(0)}\cdot(\theta-\omega)=2\pi n, n\in\mathbb{Z}$. For $x\in U_0$ we have

$$(2.43)\qquad \Big|\frac{mk_n}{h}(x-x^{(0)})\cdot(\theta-\omega)\Big|\le 2\frac{mk_n}{h}\varepsilon_0.$$

Therefore, choosing ε_0 small enough we get

$$(2.44)\qquad |u(x,t,\theta)-v(x,t,\omega)|^2 = 4\sin^2\frac{\alpha}{2}+O(\varepsilon).$$

This proves AB effect since the probability density $|u(x,t,\theta)-v(x,t,\omega)|^2$ changes with the magnetic flux α. Note that here we cannot distinguish between $+\alpha$ and $-\alpha$ modulo $2\pi n$.

Now consider the case of several obstacles $\Omega_j,\ 1\le j\le r$ for $n=2$. Let $\alpha_j=\frac{e}{hc}\int_{\gamma_j}A(x)\cdot dx,\ 1\le j\le r$, where γ_j is a simple contour encircling Ω_j only. Let $x^{(1)}\notin\overline{\Omega}'=\bigcup_{j=1}^r\overline{\Omega}_j$. Denote by $\gamma^{(0)}=\gamma_0\cup\gamma_2\cup...\cup\gamma_d$ the broken ray starting at $x^{(1)}$ and reflecting at Ω' at points $x^{(2)},...,x^{(d-1)}$. Let $\omega_p,\ 1\le p\le d$, be the directions of γ_p. Note that $\omega_{p+1}=\omega_p-2(\nu(x^{(p)})\cdot\omega_p)\nu(x^{(p)})$ where $\nu(x^{(p)})$ is the outward unit normal to Ω'. The last leg γ_d of this broken ray does not intersect Ω' and can be extended to infinity. Let $x^{(0)}$ be some point on γ_d. It was proven in [21] that there exists a solution $u(x,t)$ of (2.36) satisfying boundary conditions (2.38) and such that $\operatorname{supp} u(x,t)$ is contained in a small neighborhood of the broken ray $\gamma^{(0)}$ and $u(x,t)$ has the following form in a small neighborhood of the point $x^{(0)}$:

(2.45)

$$u(x,t)=c_0(x,t')\exp i\Big(-\frac{mk^2t}{2h}+\frac{mk}{h}\psi_d(x)+\frac{e}{hc}\int\limits_{\gamma(x(t'))}A(x)\cdot dx\Big)+O\Big(\frac{1}{k}\Big),$$

where

$$\Big|\frac{\partial\psi_d}{\partial x}\Big|^2=1,\quad \frac{\partial\psi_d(x^{(0)})}{\partial x}=\omega_d,\quad c_0(x,t')\ne 0,\quad t=\frac{t'}{k}.$$

In (2.45) we denoted by $\gamma(x(t'))$ the broken ray starting near $x^{(1)}$ at $t=0$ and ending at $x(t')$ near $x^{(0)}$. In particular, $\gamma(x(t^{(0)})=\gamma(x^{(0)})=\gamma^{(0)}$, i.e. $x(t^{(0)})=x^{(0)}$ is the endpoint of $\gamma^{(0)}$. We assume that ω_1 is the direction of the first leg of all broken rays starting near $x^{(1)}$ at $t=0$. We choose the endpoints of $\gamma^{(0)},\ x^{(1)}$ and $x^{(0)}$, far from the obstacles. Thus, the straight ray β starting at $x^{(1)}$ and ending at $x^{(0)}$ does not intersect the obstacles. If $x=\hat{x}(t)$ is the equation of β, where t is the

time parameter, we assume that $\hat{x}(t_1) = x^{(1)}$ and $\hat{x}(t^{(0)}) = x^{(0)}$. Thus t_1 is equal to $t^{(0)} - |\beta|$ where $|\beta|$ is the length of β (cf Fig.2). We can construct a solution $v(x,t)$ such that

(2.46)
$$\begin{aligned} v(x,t) =&\chi_0\Big(\frac{(x-x^{(2)})\cdot\theta_\perp}{\delta_1}\Big)c_1(x,t') \\ &\cdot\exp\Big(-i\frac{mk^2t}{2h}+\frac{imk}{h}x\cdot\theta+\frac{ie}{hc}\int\limits_{\beta(x(t'))}A(x)\cdot dx+O\Big(\frac{1}{k}\Big), \end{aligned}$$

where $t = \frac{t'}{k}$, $(x,t') \in U_0$ where U_0 is a neighborhood of $(x^{(0)}, t^{(0)})$.

We choose the initial condition $c_1(x,t_1)$ such that $c_1(x,t') = c_0(x,t')$ at $(x^{(0)}, t^{(0)})$.

As in (2.42) we have (cf. [21])

(2.47)
$$\begin{aligned} &|u(x,t)-v(x,t)|^2 \\ =&|c_0(x^{(0)},t^{(0)})|^2\Big(4\sin^2\frac{1}{2}\Big(\frac{mk}{h}(\psi_d(x)-\theta\cdot x)\Big)+I_1-I_2\Big)+O(\varepsilon) \\ =&|c_0(x^{(0)},t^{(0)}|^2 4\sin^2\frac{1}{2}(I_1-I_2)+O(\varepsilon), \end{aligned}$$

where I_1 and I_2 are integrals of $A(x)$ over $\gamma(x^{(0)})$ and $\beta(x^{(0)})$, respectively.

Note that $I_1 - I_2 = \alpha_\gamma$ where α_γ is the sum of magnetic fluxes of all obstacles encircled by $\gamma^{(0)}$ and β.

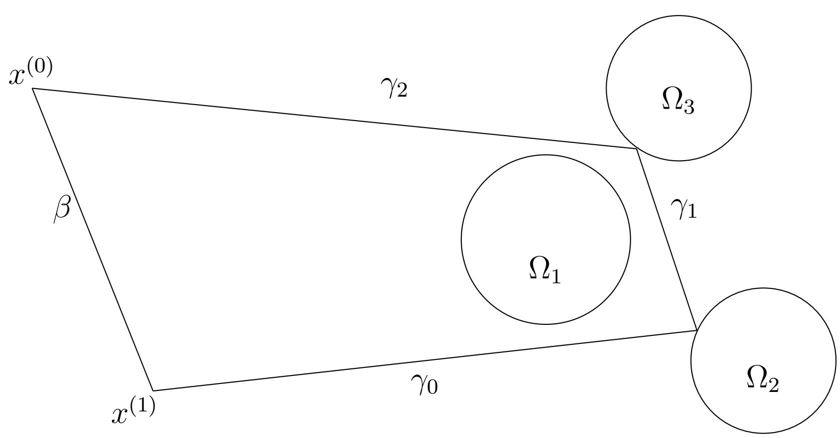

Fig. 2. Broken ray $\gamma^{(0)} = \gamma_0 \cup \gamma_1 \cup \gamma_2$ starts at $x^{(1)}$ at $t = 0$, reflects at Ω_2 and Ω_3 and ends at $x^{(0)}$ at $t = t^{(0)}$. The ray β starts at $x^{(1)}$ at $t = t_1$ and ends at $x^{(0)}$ at $t = t^{(0)}$.

Varying $\gamma^{(0)}$ and β at least r times we get enough linear relations to recover two gauge equivalence classes: $\{\alpha_j \,(\mathrm{mod}\ 2\pi n),\ 1 \le j \le r\}$ and $\{-\alpha_j \,(\mathrm{mod}\ 2\pi n), 1 \le j \le r\}$.

3. Electric AB effect

In this section we shall study the electric AB effect. Consider the Schrödinger equation with electric potential $V(x,t)$ and zero magnetic potential in $(\mathbb{R}^n \times [0,T]) \setminus \Omega$ where Ω is a domain in $\mathbb{R}^n \times [0,T]$ that we shall describe below. We have

$$ih\frac{\partial u(x,t)}{\partial t} + \frac{h^2}{2m}\Delta u(x,t) - eV(x,t)u(x,t) = 0 \tag{3.1}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n \setminus \Omega_0, \tag{3.2}$$

$$u\Big|_{\partial\Omega_{t_0}} = 0, \quad 0 < t_0 < T, \tag{3.3}$$

where $\Omega_{t_0} = \Omega \cap \{t = t_0\}$. We assume that the normals to Ω in $\mathbb{R}^n \times (0,T)$ are not parallel to the t-axis when $0 < t < T$.

Consider the following domain: Suppose $\Omega(\tau)$ in polar coordinates (r,θ) has the following form (cf. Fig.3):

$$\Omega(\tau) = \{(r,\theta)\ s.t.\ R_1 < r < R_2,\ -\pi + \tau \le \theta \le \pi - \tau\},$$

where $0 \le \tau < \pi$. Note that $\Omega(0)$ is the annulus domain $\{R_1 < |x| < R_2\}$. Let $\Omega_{t_0} = \Omega \cap \{t = t_0\}$ be equal to $\Omega(\varepsilon - t_0)$ when $0 \le t_0 \le \varepsilon$, $\Omega_{t_0} = \Omega(0)$ for $\varepsilon \le t_0 \le T - \varepsilon$, $\Omega_{t_0} = \Omega(t_0 - T + \varepsilon)$ for $T - \varepsilon \le t_0 \le T$. Thus $\Omega = \bigcup_{0 \le t \le T} \Omega_t$ is a time-dependent obstacle.

Let $D = (\mathbb{R}^n \times (0,T)) \setminus \Omega, \mathcal{D}_{t_0} = D \cap \{t = t_0\}$. The domains D_{t_0} are connected when $0 \le t_0 < \varepsilon$ and $T - \varepsilon < t_0 \le T$ and they are not connected when $\varepsilon \le t_0 < T - \varepsilon$: there are two connected components: $|x| > R_2$ and $|x| < R_1$ for $\varepsilon \le t_0 \le T - \varepsilon$.

We consider two Schrödinger equations in D of the form (3.1). The first is when $V_1(x,t) \equiv 0$ in D and the second is when $V_2(x,t) = 0$ outside $Q = \{(x,t) : R_1 \le |x| \le R_2, \varepsilon \le t \le T - \varepsilon\}$ and $V_2(x,t) = V_2(t)$ in Q. Note that $E = \frac{\partial V_2(x,t)}{\partial x} = 0$ in D. We choose $V_2(t)$ such that

$$\frac{e}{h}\int_{\varepsilon}^{T-\varepsilon} V_2(t)dt = \alpha \neq 2\pi p,\ \forall p \in \mathbb{Z},\ \ V_2(t) = 0 \text{ near } t = \varepsilon \text{ and } t = T - \varepsilon. \tag{3.4}$$

We assume that $u_1(x,t)$ and $u_2(x,t)$ have the same initial and boundary conditions where u_1 corresponds to $V_1 = 0$ and u_2 corresponds to $V_2(x,t)$.

We shall prove that $|u_1(x,t)| \not\equiv |u_2(x,t)|$ for $t > T - \varepsilon$ when (3.4) holds, i.e. electric AB effect takes place.

When $0 < t < \varepsilon$ we have that $u_1(x,t) = u_2(x,t)$ since $V_1 = V_2 = 0$ for $(x,t) \in D,\ t < \varepsilon$. For $\varepsilon < t < T - \varepsilon$ we have that

$$u_2(x,t) = \Big(\exp \frac{ie}{h}\int_\varepsilon^t V_2(t')dt\Big)u_1(x,t)$$

for $(x,t) \in Q,\ \varepsilon < t \le T - \varepsilon$.

Let $x^{(2)}$ be such that $R_1 < |x^{(2)}| < R_2$ and $u_1(x^{(2)}, T - \varepsilon) \neq 0$ and let $|x^{(1)}| > R_2$ be such that $u_1(x^{(1)}, T - \varepsilon) \neq 0$.

We can choose the initial condition $u_0(x)$ such that this holds. Indeed, let $u_1(x)$ be any function such that $u_1(x^{(1)}) \neq 0$ and $u_1(x^{(2)}) \neq 0$. Consider the backward initial boundary value problem for $0 < t < T - \varepsilon,\ ih\frac{\partial u_1}{\partial t} + \frac{h^2}{2m}\Delta u_1 = 0$ for $(x,t) \in D,\ u_1(x, T-\varepsilon) = u_1(x),\ u_1\big|_{\partial\Omega} = 0$.

Then we take $u_0(x) = u_1(x,0)$ as the initial condition for the initial boundary value problem (3.1), (3.2), (3.3).

We claim (cf. [21]) that $|u_1(x,t)|^2 \neq |u_2(x,t)|^2$ in a neighborhood U_0 of $(x_1^{(0)}, T - \varepsilon)$ for $t > T - \varepsilon$. Note that $u_1(x, T-\varepsilon) = u_2(x, T-\varepsilon)$ for $|x| > R_2$ since $u_1(x,\varepsilon) = u_2(x,\varepsilon)$ for $|x| > R_2$ and u_1, u_2 have zero boundary conditions on $\{(|x| = R_2) \times (\varepsilon, T-\varepsilon)\}$. In particular $u_1(x, T-\varepsilon) = u_2(x, T-\varepsilon)$ in U_0. Suppose that $|u_1(x,t)| = |u_2(x,t)|$ in U_0 for $t > T - \varepsilon$.

Use the polar representation in $U_0 \cap \{t > T - \varepsilon\}$:

$$u_1(x,t) = R_1(x,t)e^{i\Phi_1(x,t)},\ u_2(x,t) = R_2(x,t)e^{i\Phi_2(x,t)}.$$

Note that $R_1 = R_2 = R$.

Substituting in (3.1) we get

$$-h\frac{\partial R}{\partial t} = \frac{h^2}{2m}(2\nabla R \cdot \nabla\Phi_j + R\Delta\Phi_j), \tag{3.5}$$

$$h\frac{\partial \Phi_j}{\partial t}R = \frac{h^2}{2m}(\Delta R - R|\nabla\Phi_j|^2), \quad j = 1, 2. \tag{3.6}$$

Therefore Φ_1 and Φ_2 satisfy the same first order partial differential equation (3.6) with the same initial condition in $U_0 \cap \{t = T - \varepsilon\}$.

$$\Phi_1(x, T-\varepsilon) = \Phi_2(x.T-\varepsilon)$$

since $u_1(x, T-\varepsilon) = u_2(x, T-\varepsilon)$. Therefore, by the uniqueness of the Cauchy problem we have $\Phi_1(x,t) = \Phi_2(x,t)$ in $U_0 \cap \{t > T - \varepsilon\}$. Thus $u_1(x,t) = u_2(x,t)$ in $U_0 \cap \{t > T - \varepsilon\}$. Then by the unique continuation

property for the Schrödinger equation (cf. [32], Sect. 6) we get that $u_1(x,t) = u_2(x,t)$ for $(x,t) \in D,\ T-\varepsilon < t < T$. By the continuity in t we conclude that $u_1(x, T-\varepsilon) = u_2(x, T-\varepsilon),\ R_1 < |x| < R_2$. Since $u_1(x^{(2)}, T-\varepsilon) \neq 0$ and $u_2(x^{(2)}, T-\varepsilon) = u_1(x^{(2)}, T-\varepsilon) \exp i\alpha$ we got a contradiction since $\exp i\alpha \neq 1$ when α satisfies (3.4). This concludes the proof of electric AB effect.

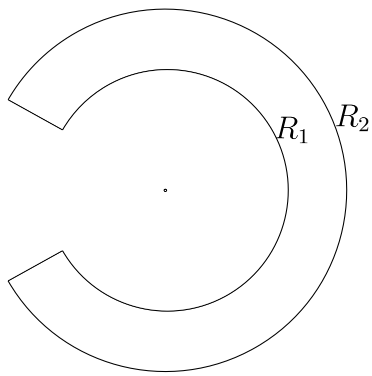

Fig. 3. The intersection D_{t_0} of the domain D with the plane $t = t_0$ is the complement in $\mathbb{R}^2$ of $\Omega(\tau) = \{R_1 \leq |x| \leq R_2,\ -\pi+\tau \leq \theta \leq \pi-\tau\}$, where τ depends on t_0. When $\tau = 0$ D_{t_0} has two connected components.

4. The Schrödinger equation with time-dependent magnetic and electric potentials

The case of time-dependent electromagnetic potentials is much harder than the case when A and V are time-independent. Many powerful tools such as the BC-method are not applicable. Therefore the results on the inverse boundary value problems are much weaker. The study of the Aharonov-Bohm effect also becomes more complicated.

4.1. **Inverse boundary value problem.**

Let $\Omega_j(t) \subset \mathbb{R}^n, 0 \leq t \leq T$, be the obstacles, $\overline{\Omega}_j(t) \cap \overline{\Omega}_k(t) = 0, 1 \leq j,k \leq r$, and let $\Omega_0 \supset \overline{\Omega}' = \bigcup_{k=1}^n \overline{\Omega}_j(t)$, where Ω_0 is a simply-connected domain in $\mathbb{R}^n$. Let $\Omega' = \bigcup_{0\leq t\leq T} \bigcup_{g=1}^r \Omega_j(t)$. Consider in $(\Omega_0 \times (0,T)) \setminus \overline{\Omega}'$ the Schrödinger equation with time-dependent magnetic and electric potentials

$$
(4.1) \quad ih\frac{\partial u(x,t)}{\partial t} - \frac{1}{2m}\sum_{j=1}^n \Big(-ih\frac{\partial}{\partial x_j} - \frac{e}{c}A_j(x,t)\Big)^2 u(x,t) \\
- eV(x,t)u(x,t) = 0, \quad (x,t) \in (\Omega_0 \times (0,T)) \setminus \Omega'.
$$

We assume that

$$u\Big|_{\partial\Omega_j(t)} = 0, \quad t \in (0,T), \quad j = 1, ..., r. \tag{4.2}$$

We also assume that the normals to Ω' for $0 < t < T$ are not parallel to the t-axis. This condition assures the existence of the solution of the initial-boundary value problem for (4.1).

The gauge group $G((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$ in this case consists of $g(x,t) \in C^\infty((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$ such that $|g(x,t)| = 1$. Thus the electromagnetic potentials $(A(x,t), V(x,t))$ and $(A'(x,t), V'(x,t))$ are gauge equivalent if there exists $g(x,t) \in G((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$ such that

$$\begin{aligned} \frac{e}{c}A'(x,t) &= \frac{e}{c}A(x,t) + ihg^{-1}(x,t)\frac{\partial g(x,t)}{\partial x}, \\ eV'(x,t) &= eV(x,t) - ihg^{-1}(x,t)\frac{\partial g(x,t)}{\partial x}. \end{aligned} \tag{4.3}$$

Now we shall describe the class of obstacles considered in this subsection. Since the potential depends on the time variable we cannot switch from the Schrödinger equation to the wave equation and use the Boundary Control method as in section 2.1. We shall use instead the inversion of the X-ray transform and this approach in the presence of obstacles imposes severe restrictions on the obstacles.

In the case of $n \geq 3$ variables we assume that the following condition is satisfied

(4.4) For each $t_0 \in [0,T]$ all obstacles $\Omega_j(t_0)$, $1 \leq j \leq r$, are convex, and for each point $x_0 \in \Omega_0 \setminus \bigcup_{k=1}^{r} \overline{\Omega}_j(t_0)$ there exists a two dimensional plane $\Pi_{x_0} \subset \mathbb{R}^n, n \geq 3$ that intersect at most one of the obstacles $\Omega_j(t_0)$.

In the case of $n = 2$ we assume that

(4.5) All obstacles $\Omega_j(t_0)$ are convex in $\mathbb{R}^2$ for each $t_0 \in [0,T]$, $1 \leq j \leq r$. If $r > 1$, i.e. when there are more then one obstacle, we assume that there is no trapped broken (reflected) rays in $\Omega_0 \setminus \bigcup_{j=1}^{r} \Omega_j(t_0)$, i.e. any broken ray starting on $\partial\Omega_0$ returns to $\partial\Omega_0$ after a finite number of reflections.

When the obstacles are smooth and $r \geq 2$ there are always trapped rays. To have the situation when there are no trapped rays we must require that obstacles $\Omega_j(t_0),\ 1 \leq j \leq r$, have a finite number of corner points.

We consider only the broken rays avoiding corners points, and we assume that the number of reflections is uniformly bounded for all broken rays.

As in §2.3 we introduce gauge invariant boundary data on $\partial\Omega_0 \times (0,T)$:

(4.6)

$$|u(x,t)|^2 = f_1,\ \frac{\partial}{\partial\nu}|u(x,t)|^2 = f_2,\ S(u) = \Im\Big(h\frac{\partial u}{\partial x} - i\frac{e}{c}A(x,t)u\Big)\overline{u} = f_3,$$

where $(x,t) \in \partial\Omega_0 \times (0,T)$.

The following theorem holds (cf. [12], [13], [18]).

Theorem 4.1. *Consider two Schrödinger equations* $\big(ih\frac{\partial}{\partial t} - H\big)u = 0$ *and* $\big(ih\frac{\partial}{\partial t} - H'\big)u' = 0$ *of the form (4.1) in* $\Omega_0 \times (0,T) \setminus \Omega'$ *with zero Dirichlet boundary conditions on* $\partial\Omega'$ *and zero initial conditions on* $\Omega_0 \setminus \bigcup_{j=1}^r \Omega_j(0)$*, corresponding to electromagetic potentials* $(A(x,t), V(x,t))$ *and* $(A'(x,t), V'(x,t))$*, respectively.*

Suppose obstacles Ω' *satisfy condition (4.4) when* $n \geq 3$ *and the condition (4.5) when* $n = 2$*.*

If the sets of gauge invariant boundary data of u *and* u' *are equal on* $\partial\Omega_0 \times (0,T)$ *then the electromagnetic potentials* (A, V) *and* (A', V') *are gauge equivalent.*

The proof of Theorem 4.1 was given in [18]. Since the case of time-dependent potentials is relatively new we shall indicate the main steps of the proof.

Proof: It was shown in §2.3 that the equality of the gauge invariant boundary data is equivalent to the existence of $g_0 \in G((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$ such that the corresponding DN operators Λ and Λ' are gauge equivalent on $\partial\Omega_0 \times (0,T)$, i.e. $\Lambda' v = g_{00}^{-1}\Lambda g_{00} v$ for any smooth v on $\partial\Omega_0 \times (0,T)$. Here g_{00} is the restriction of $g_0(x,t)$ to $\partial\Omega_0 \times (0,T)$.

Making the gauge transformation $w = g_0^{-1}u''$ we get the Schrödinger equation $\big(ih\frac{\partial}{\partial t} - H''\big)w = 0$ with electromagnetic potentials $(A''.V'')$ that are gauge equivalent to (A', V'). Now we have that $\Lambda = \Lambda''$ on $\partial\Omega_0 \times (0,T)$ where Λ'' is the DN operator corresponding $\big(ih\frac{\partial}{\partial t} - H''\big)w = 0$.

Consider first the more simple case of $n \geq 3$ assuming that (4.4) holds.

Let $\gamma(t_0)$ be a ray in the domain $\Omega_0\setminus\bigcup_{j=1}^r\Omega_j(t_0)$ starting and ending on $\partial\Omega_0$. We shall construct a solution $u(x,t,k)$ of $\big(ih\frac{\partial}{\partial t}-H\big)u=0$ in $(\Omega_0\times(0,T)\setminus\Omega')$ depending on a large parameter k and satisfying the boundary conditions

$$u\Big|_{\partial\Omega'}=0,\quad 1\le j\le r, \tag{4.7}$$

initial conditions

$$u\Big|_{\Omega_0\setminus\Omega'\cap\{t=0\}}=0, \tag{4.8}$$

and concentrated in a small neighborhood $U_0\subset(\Omega_0\times(0,T))\setminus\Omega'$ of the ray $\gamma(t_0)$.

We are looking for $u(x,t,k)$ in the form

$$u(x,t,k)=e^{-i\frac{mk^2}{2h}t+i\frac{mk}{h}x\cdot\omega}\Big(\sum_{p=0}^N\frac{a_{p0}(x,t,\omega)}{(ik)^p}+O\Big(\frac{1}{k^{N+1}}\Big)\Big), \tag{4.9}$$

where

$$\begin{aligned}&|\omega|=1,\\ &\omega\cdot\Big(-ih\frac{\partial}{\partial x}-\frac{e}{c}A(x)\Big)a_{00}=0,\\ &\omega\cdot\Big(-\frac{ih\partial}{\partial x}-\frac{e}{c}A(x)\Big)a_{p0}=\Big(ih\frac{\partial}{\partial t}-H\Big)a_{p-1,0},\quad p\ge 1.\end{aligned} \tag{4.10}$$

We choose

$$\begin{aligned}a_{00}={}&\frac{1}{\varepsilon^{\frac{n}{2}}}\chi_0\Big(\frac{t-t_0}{\varepsilon}\Big)\prod_{j=1}^{n-1}\chi_0\Big(\frac{\tau_j-\tau_{0j}}{\varepsilon}\Big)\\ &\cdot\exp\Big(i\frac{e}{hc}\int_{s_0}^{s}A\Big(\sum_{j=1}^{n-1}\tau_j\omega_{\perp j}+s'\omega,t\Big)\cdot\omega ds',\end{aligned} \tag{4.11}$$

where $s=x\cdot\omega,\ \tau_j=x\cdot\omega_{\perp j},\ \chi_0$ is the same as in (2.39), $\int_{-\infty}^{\infty}\chi_0^2(s)ds=1,\ s_0,r_{0j}$ are such that $x^{(0)}=s_0\omega+\sum_{j=1}^{n-1}\tau_{0j}\omega_{\perp j}\notin\Omega_0$, where $\omega\cdot\omega_{\perp j}=0, 1\le j\le n-1,\ \{\omega,\omega_{\perp 1}...,\omega_{\perp,n-1}\}$ is an orthogonal basis in $\mathbb{R}^n$. Also the plane $(x-x^{(0)})\cdot\omega=0$ does not intersect Ω_0 (cf. [13], [18]). We assume also that

$$a_{p0}(s,\tau,t)=0\ \text{ when }\ s=s_0,\ p\ge 1. \tag{4.12}$$

Note that the principal term (4.10) of (4.9) is the same as in the case of potentials independent of t. However, the lower other terms $a_{p0},\ p\ge 1$, will pick up the derivatives of A,V in t.

Solution of the form (4.9) is different from the geometric optics type solutions in §2.6. The latter solutions describe the propagation in the time and (4.9) propagates in the plane $t = t_0$ along the space direction $s = x \cdot \omega$.

We shall show below that solutions (4.9) can be approximated by physically relevant solutions.

Having solutions of the form (4.9) we can conclude the proof of Theorem 4.1 in two steps. First, substituting the solutions of $\big(ih\frac{\partial}{\partial t} - H\big)u = 0$ and $\big(ih\frac{\partial}{\partial t} - H''\big)w = 0$ having both the form (4.9), in the Green's formula, using that $\Lambda = \Lambda''$ on $\partial\Omega_0 \times (0,T)$ and passing to the limit when $\varepsilon \to 0$ we get

$$\exp\Big(\frac{ie}{hc}\int\limits_{\gamma(t_0)} A(x,t_0)\cdot dx\Big) = \exp\Big(\frac{ie}{hc}\int\limits_{\gamma(t_0)} A''(x,t_0)\cdot dx\Big) \tag{4.13}$$

for all rays $\gamma(t_0)$, $t_0 \in (0,T)$ is arbitrary, but fixed. Now, using the Helgason's hole theorem (cf. [Hel]), we prove the uniqueness of the X-ray transform to get that there exists $g(x,t) = e^{i\varphi(x,t)} \in G((\overline{\Omega}_0 \times [0,T]) \setminus \Omega')$, $g = 1$ on $\partial\Omega_0 \times (0,T)$, such that

$$\frac{e}{c}A''(x,t_0) = \frac{e}{c}A(x,t_0) + ihg^{-1}\frac{\partial g(x,t_0)}{\partial x}.$$

Here t_0 is a parameter. Making the gauge transformation in $\big(ih\frac{\partial}{\partial t} - H''\big)w = 0$ with the gauge $g(x,t)$ we get the Schrödinger equation $\big(ih\frac{\partial}{\partial t} - H'''\big)w_1 = 0$ with magnetic potential $A(x,t)$ and the electric potential $eV''' \equiv eV'' - ihg^{-1}\frac{\partial g}{\partial t}$. Now apply again the Green's formula to $\big(ih\frac{\partial}{\partial t} - H\big)u = 0$ and $\big(ih\frac{\partial}{\partial t} - H'''\big)w = 0$ using the solution of the form (4.9) and that $\Lambda = \Lambda'''$. Since H' and H''' have the same magnetic potentials, their contribution will cancel each other and we get that

$$\int\limits_{\gamma(t_0)} \Big(eV(x,t_0) - eV'' + ihg^{-1}\frac{\partial g}{\partial t}\Big)ds = 0 \tag{4.14}$$

for all rays $\gamma(t_0)$.

Therefore, the uniqueness theorem of the X-ray transform gives that

$$eV(x,t) - eV'''(x,t) + ihg^{-1}(x,t)\frac{\partial g}{\partial t} = 0.$$

Thus (A,V) and (A''',V''') are gauge equivalent. Hence (A,V) and (A',V') are gauge equivalent too.

Now consider a more difficult case $n = 2$ and (4.5) is satisfied. As in §2.6 we will use the broken rays.

Let $\gamma(t_0) = \gamma_0(t_0) \cup \gamma_1(t_0) \cup ... \cup \gamma_d(t_0)$ be a broken ray starting at some point $x^{(0)} \in \partial\Omega_0$ reflecting at some obstacles $\Omega_j(t_0), 1 \leq j \leq r$, and ending on $\partial\Omega_0$. We shall construct a solution of $\left(i\frac{\partial}{\partial t} - H\right)u = 0$ concentrated in a small neighborhood of $\gamma(t_0)$. We are looking for $u(x,t,k)$ in the form

$$u(x,t,k) = \sum_{j=0}^{d}\sum_{p=0}^{N} \frac{a_{pj}(x,t.\omega)}{(ik)^p} e^{-\frac{imk^2}{2h}t + i\frac{mk}{h}\psi_j(x,t,\omega)} \tag{4.15}$$

where $\psi_0(x,t_0,\omega) = x\cdot\omega,\ a_{p0}(x,t,\omega)$ are the same as in (4.9), $\omega = \theta_1$ is the direction of γ_0, $\psi_j(x,t,\omega)$ satisfy the equations

$$\begin{aligned} &\left|\frac{\partial\psi_j}{\partial x}\right| = 1, \quad \psi_j(x,t,\omega)\Big|_{\partial\Omega'} = \psi_{j+1}(x,t,\omega)\Big|_{\partial\Omega'}, \\ &\frac{\psi_{j+1}(x_0^{j+1},t_0,\omega)}{\partial x} = \theta_{j+1}, \quad 0 \leq j \leq d-1, \end{aligned} \tag{4.16}$$

where $x_0^{(j+1)}$ is the point of reflection of γ_j at $\partial\Omega' \cap \{t = t_0\}$ and θ_{j+1} is the direction of γ_{j+1}.

Functions a_{pj} satisfy the following equations:

$$\begin{aligned} &\frac{\partial a_{pj}}{\partial x}\cdot\frac{\partial\psi_j}{\partial x} + \frac{1}{2}\Delta\psi_j a_{pj} - i\frac{e}{c}A(x,t)\cdot\frac{\partial\psi_j}{\partial x}a_{pj} \\ &\qquad = f_{pj}(x,t,\omega) + i\frac{m}{h}\frac{\partial\psi_j}{\partial t}a_{pj}, \quad p \geq 0, \end{aligned} \tag{4.17}$$

where $f_{0j} = 0, f_{pj}$ depends on $a_{0j}, ..., a_{p-1,j}$.

When obstacles are independent of t then $\psi_j(x,t)$ are also independent of t. We impose also the following conditions on a_{pj}

$$a_{pj}\big|_{\partial\Omega'} = -a_{p,j+1}\big|_{\partial\Omega'} \tag{4.18}$$

This last condition implies that $u\big|_{\partial\Omega'} = 0$. Inserting (4.15) into the Green's formula instead of (4.9) we get, analogously to (4.13), (4.14), that

$$\exp\Big[\frac{ie}{hc}\sum_{j=0}^{d}\int_{\gamma_j(t_0)} (A(x_0^{(j)}+s\theta_j,t_0) - A''(x_0^{(j)}+s\theta_j,t_0))\cdot\theta_j ds\Big] = 1 \tag{4.19}$$

and

(4.20)

$$\sum_{j=0}^{d}\int_{\gamma_j(t_0)} \Big(eV(x_0^{(j)}+s\theta_j,t_0) - eV'''(x_0^{(j)}+s\theta_j,t_0) + ihg^{-1}\frac{\partial g}{\partial t}\Big)ds = 0.$$

Proving the uniqueness of X-ray transform problem for broken rays is much harder. It was shown in [13], [15] that (4.19), (4.20) imply that the electromagnetic potentials (A, V) and (A', V') are gauge equivalent.

This concludes the proof of Theorem 4.1.

4.2. **Inverse boundary value problems for the Schrödinger operator with time-dependent Yang-Mills potentials.**
Consider the Schrödinger equation of the form
(4.21)

$$-i\frac{\partial u(x,t)}{\partial t}+\sum_{j=1}^{n}\Big(I_m\Big(-i\frac{\partial}{\partial x_j}\Big)-A_j(x,t)\Big)u(x,t)+V(x,t)u(x,t)=0,$$

where $A=(A_1,...,A_n),\ A_j(x,t), V(x,t), 1\leq j\leq n$, are $m\times m$ self-adjoint matrices. It is convenient to consider $u(x,t)$ also as $m\times m$ matrix. I_m is $m\times m$ identity matrix.

We consider (4.21) in $\Omega_0\times(0,T)$ with initial conditions

$$u(x,0)=0,\quad x\in\Omega_0,$$

and the boundary conditions

$$u\Big|_{\partial\Omega_0\times(0,T)}=f.$$

Let $\Lambda f=\big(I_m\frac{\partial}{\partial\nu}-iA\cdot\nu\big)u\big|_{\partial\Omega_0\times(0,T)}$ be the DN operator. The gauge group consists of $m\times m$ unitary matrices, smooth in $\overline{\Omega}_0\times[0,T]$. We assume that there is no obstacles in this subsection.

Yang-Mills potentials (A,V) and (A',V') are gauge equivalent if there is $g\in G(\overline{\Omega}_0\times[0,T])$ such that

$$\begin{aligned}A_j'&=g^{-1}A_jg+ig^{-1}\frac{\partial g}{\partial x_j},\quad 1\leq j\leq n\\ V_j'&=g^{-1}V_jg-ig\frac{\partial g}{\partial t}\end{aligned}\tag{4.22}$$

Theorem 4.2. *Consider two equations* $\big(-i\frac{\partial}{\partial t}+H\big)u=0,\ \big(-i\frac{\partial}{\partial t}+H'\big)u'=0$ *of the form (4.21) with Yang-Mills potentials* $(A,V),(A',V')$, *respectively. Suppose that DN operators* Λ *and* Λ', *corresponding to* $\big(-i\frac{\partial}{\partial t}+H\big)u=0$ *and* $\big(-i\frac{\partial}{\partial t}+H'\big)u'=0$ *are gauge equivalent on* $\partial\Omega_0\times(0,T)$ *with some gauge* $g_0(x)$, *i.e.* $g_{00}^{-1}\Lambda g_{00}=\Lambda'$, *where* $g_{00}=g_0\big|_{\partial\Omega_0\times(0,T)}$. *Then* (A,V) *and* (A',V') *are gauge equivalent too.*

The beginning of the proof of Theorem 4.2 is similar to the proof of Theorem 4.1 in the case $n\geq 3$.

We construct a solution of (4.21) similar to (4.9)

$$u_\varepsilon(x,t,k) = e^{-ik^2t+ikx\cdot\omega}\Big(\sum_{p=0}^{N}\frac{a_p(x,t,\omega)}{(ik)^p} + O\Big(\frac{1}{k^{N+1}}\Big)\Big), \tag{4.23}$$

where

$$a_0(x,t,\omega) = \frac{1}{\varepsilon^{\frac{n}{2}}}\chi_0\Big(\frac{t-t_0}{\varepsilon}\Big)\Pi_{j=1}^{n-1}\chi_0\Big(\frac{\tau_j-\tau_{j0}}{\varepsilon}\Big)c(x,t,\omega),$$

t_0, τ_{j0}, τ_j are the same as in (4.11), $c(x,t,\omega)$ satisfies the equation

$$\omega\cdot\frac{\partial c}{\partial x} - iA(x,t)\cdot\omega c = 0 \quad \text{for} \quad s > s_0, \quad c = I_m \quad \text{when} \quad s = s_0, \tag{4.24}$$

a_p satisfy equations similar to (4.10) and $a_p(s,\tau,t) = 0$ when $s = s_0,\ p \geq 1$.

Applying gauge g_0 to $\big(-i\frac{\partial}{\partial t} + H'\big)u' = 0$ we get an equation $\big(-i\frac{\partial}{\partial t} + H''\big)u'' = 0$, gauge equivalent to $\big(-i\frac{\partial}{\partial t} + H'\big)u' = 0$ and such that $\Lambda'' = \Lambda$.

Using the Green's formula and passing the limit as $\varepsilon \to 0$ we get, similarly to Theorem 4.1 that

$$c_0(+\infty, y', t_0, \omega) = c_0''(+\infty, y', t_0, \omega), \tag{4.25}$$

where $y_1 = x\cdot\omega,\ y' = x-(x\cdot\omega)\omega,\ c_0(y_1,y',t_0,\omega)$ and $c_0''(y_1,y',t_0,\omega)$ are matrices $c(x,t_0,\omega)$ and $c''(x,t_0,\omega)$ (cf. (4.24)) in (y_1,y') coordinates, c corresponds to $\big(-i\frac{\partial}{\partial t}+H\big)u = 0,\ \ c''$ corresponds to $\big(-i\frac{\partial}{\partial t}+H''\big)u'' = 0$.

Note that (4.25) is the analog of (4.13) when $m > 1$.

The matrix $c_0(+\infty, y', t_0, \omega)$ is called the non-Abelian Radon transform of $A(x)$. The problem of the recovery of $A(x)$ from the non-Abelian Radon transform is much more difficult then in the case of electromagnetic potentials, i.e. when $m = 1$. This was done in [11], [14], [40]. The recovery of $V(x,t)$ was also done in [11], [14]]. Note that the most difficult case is $n = 2$. The extension to $n \geq 3$ dimensions is relatively easy (cf. [18]).

4.3. An inverse problem for the time-dependent Schrödinger equation in an unbounded domain.

When the Schrödinger operator with time-independent coefficients is studied in $\mathbb{R}^n$ outside the obstacles, it is natural to consider the scattering problem. When the coefficients are time-dependent we propose a new problem.

Consider the Schrödinger equation of the form

$$-ih\frac{\partial u}{\partial t} + Hu = 0 \quad \text{in} \quad (\mathbb{R}^n\times(0,T))\setminus\Omega', \tag{4.26}$$

where

$$Hu = \frac{1}{2m}\sum_{j=1}^{n}\Big(-ih\frac{\partial}{\partial x_j} - \frac{e}{c}A(x,t)\Big)^2 u(x,t) + eV(x,t)u(x,t),$$

$\Omega' = \bigcup\limits_{0\le t\le T}\bigcup\limits_{j=1}^{r}\Omega_j(t)$ are obstacles. We assume that the electromagnetic potentials are independent of t for $|x|\ge R$ where R is such that $\Omega'\subset B_R = \{|x|<R\}$. We assume also that $V(x) = O\big(\frac{1}{|x|^{1+\varepsilon}}\big)$, $A(x) = O\big(\frac{1}{|x|}\big)$. Here $V(x) = V(x,t)$, $A(x) = A(x,t)$ for $|x|>R, t\in[0,T]$.

Assume that $u(x,t)$ satisfies the initial condition on $\mathbb{R}^n\setminus\bigcup_{j=1}^{r}\Omega_j(0)$:

$$u(x,0) = u_0(x), \quad u_0(x) = 0 \ \text{ on } \ B_R\setminus(\Omega'\cap\{t=0\}). \tag{4.27}$$

The gauge group $G((\mathbb{R}^n\times[0,T])\setminus\Omega')$ is different in the cases $n=2$ and $n\ge 3$.

We assume that $|g(x,t)| = 1$ in $(\mathbb{R}^n\times[0,T])\setminus\Omega'$ and $g(x,t) = e^{i\frac{\varphi(x)}{h}}$, $\varphi(x) = O\big(\frac{1}{|x|}\big)$ for $n\ge 3$. When $n=2$ we assume $g(x,t) = e^{ip\theta(x)}\big(1+O\big(\frac{1}{|x|}\big)\big)$ where $p\in\mathbb{Z}$ and $\theta(x)$ is the polar angle. We also assume that the origin belongs to $\Omega'\cap\{t=0\}$.

Suppose we are given initial conditions for the equation (4.26) for $t=0$ and the condition

$$u(x,T) = u_1(x), \quad |x|>R \tag{4.28}$$

for $t=T$.

We shall call (4.27), (4.28) the two times data, $t=0$ and $t=T$, for the equation (4.26). We shall prove that these data determines electromagnetic potentials for $|x|<R$ up to a gauge equivalence. More precisely, the following theorem holds:

Theorem 4.3. *Consider two equations $\big(-ih\frac{\partial}{\partial t}+H\big)u = 0$ and $\big(-ih\frac{\partial}{\partial t}+H'\big)u' = 0$ of the form (4.26) in $(\mathbb{R}^n\times(0,T))\setminus\Omega'$ with electromagnetic potentials $(A(x,t),V(x,t))$ and $(A'(x,t),V'(x,t))$, respectively. Assume that (A,V) and (A',V') are independent of t for $|x|>R$.*

Suppose $(A(x),V(x))$ and $(A'(x),V'(x))$ are gauge equivalent for $|x|>R$, i.e. there exists $g_0(x)$, $|g_0(x)|=1$, such that for $|x|>R$ we have

$$\begin{aligned}&\frac{e}{c}A'(x) = \frac{e}{c}A(x) - ihg_0^{-1}(x)\frac{\partial g_0}{\partial x},\\ &V'(x) = V(x).\end{aligned} \tag{4.29}$$

Suppose that $u(x,t)$ and $u'(x,t)$ have gauge equivalent two times data

$$u(x,0) = g_0(x)u'(x,0), \quad |x| > R, \tag{4.30}$$
$$u(x,T) = g_0(x)u'(x,T) \quad \textit{for} \quad |x| > R.$$

Then the DN operators Λ and Λ' are gauge equivalent on $\partial B_R \times (0,T)$, i.e.

$$\Lambda' f = g_{00}^{-1} \Lambda g_{00} f$$

for all smooth f on $\partial B_R \times (0,T)$ and $g_{00}(x)$ is the restriction of $g_0(x)$ to $\partial B_R \times (0,T)$.

Note that combining Theorem 4.3 with Theorem 4.1 we get that (A,V) and (A',V') are gauge equivalent.

To prove Theorem 4.3 we need two lemmas.

Lemma 4.4. *Let*

$$-ih\frac{\partial w}{\partial t} + \frac{1}{2m}\sum_{j=1}^{n}\Big(-i\frac{\partial}{\partial x_j} - \frac{e}{c}A_j(x)\Big)^2 w(x,t) + eV(x)w(x,t) = 0 \tag{4.31}$$

in $(\mathbb{R}^n \setminus B_R) \times (0,T)$, where $A_j(x), V(x), 1 \le j \le n$, are independent of t,

$$\Big|\frac{\partial^k A_j(x)}{\partial x^k}\Big| \le C_k(1+|x|)^{-1-|k|}, \tag{4.32}$$
$$\Big|\frac{\partial^k V(x)}{\partial x^k}\Big| \le C_k(1+|x|)^{-1-\varepsilon-|k|}, \quad \varepsilon > 0, \ \forall k.$$

Suppose $w(x,t) \in C([0,T], L_2(\mathbb{R}^n \setminus B_R))$, i.e. $w(x,t)$ is continuous in t on $[0,T]$ with values in $L_2(\mathbb{R}^2 \setminus B_R)$.

Suppose $w(x,0) = 0$, $w(x,T) = 0$ for $x \in \mathbb{R}^n \setminus B_R$. Then $w(x,t) = 0$ in $(\mathbb{R}^n \setminus B_R) \times (0,T)$.

Proof: Extend $w(x,t)$ by zero for $t < 0$ and for $t > T$. Let $\tilde{w}(x,\xi_0)$ be the Fourier transform of $w(x,t)$ in t. Then $\tilde{w}(x,\xi_0) \in L_2(\mathbb{R}^n \setminus B_R)$ for all ξ_0 and

$$h\xi_0\tilde{w}(x,\xi_0) + H\tilde{w}(x,\xi_0) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_R.$$

It follows from the Hörmander [31] that $\tilde{w}(x,\xi_0) = 0$ in $\mathbb{R}^n \setminus B_R$ if conditions (4.32) hold.

Therefore, $w(x,t) = 0$ in $(\mathbb{R}^n \setminus B_R) \times (0,T)$.

Remark 4.1. In this paper we mostly consider the case when $B = \operatorname{curl} A = 0$ for $|x| > R$ and $V(x) = 0$ for $|x| > R$. In such case there is a simpler way to prove Lemma 4.4 without using [31].

If $n \geq 3$ and $\operatorname{curl} A = 0,\ V = 0$ for $|x| > R$, we can make a gauge transformation $g(x)$ such that $w' = g^{-1}w(x,t)$ satisfies the equation

$$\xi_0 \tilde{w}'(x,\xi_0) - \frac{h^2}{2m}\Delta \tilde{w}'(x,\xi_0) = 0 \quad \text{for} \quad |x| > R,$$

where $\tilde{w}'(x,\xi_0)$ is the Fourier transform in t. Since $\tilde{w}'(x,\xi_0) \in L_2(\mathbb{R}^n \setminus B_R)$ we have that $\tilde{w}'(x,\xi_0) = 0$ by the classical Rellich's lemma (see, for example, [20]).

When $n = 2$ and the magnetic flux $\frac{eh}{c}\int_{|x|=R} A(x)\cdot dx = \alpha \neq 0$ we can make the gauge transformation $w' = g^{-1}(x)w(x,t)$ such that $A'(x) = \frac{\alpha}{2\pi}\frac{(x_2,-x_1)}{x_1^2+x_2^2}$ is the Aharonov-Bohm potential (cf. [1]). Then making the Fourier transform in t we shall have in polar coordinates

$$\text{(4.33)} \quad h\tilde{w}'(r,\theta,\xi_0) - \frac{h^2}{2m}\Big[\frac{\partial^2 \tilde{w}'}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{w}'}{\partial r} + \frac{1}{r^2}\Big(\frac{\partial}{\partial\theta} + i\alpha\Big)^2\Big]\tilde{w}'(r,\theta,\xi_0) = 0,$$

where $\theta \in [0, 2\pi], r > R$ and

$$\int\limits_{|x|\geq R} |\tilde{w}'(r,\theta,\xi_0)|^2 r dr d\theta < \infty \quad \text{for any} \quad \xi_0 \in \mathbb{R}.$$

The general solution of (4.33) has the form (cf. [1])

$$\tilde{w}'(r,\theta,\xi_0) = \sum_{n=-\infty}^{\infty} w_n(r,\xi_0)e^{in\theta},$$

where

$$w_n(r,\xi_0) = a_n(\xi_0)J_{n+\alpha}(kr) + b_n(\xi_0)J_{-n-\alpha}(kr), \quad k = \sqrt{\frac{2m}{h}(-\xi_0)}.$$

We have

$$\text{(4.34)} \qquad \int\limits_{|x|>R} |\tilde{w}'(x,\xi_0)|^2 dx = \sum_{n=-\infty}^{\infty}\int\limits_{r>k} |w_n(r,\xi_0)|^2 r dr.$$

Using the asymptotics of the Bessel's functions we get from (4.34) that $\int_{r>R} |\tilde{w}(r,\xi_0^2)|^2 r dr < +\infty$ iff $a_n(\xi_0) = b_n(\xi_0) = 0,\ \forall n$. Therefore $\tilde{w}(x,\xi_0) = 0$ for $|x| > R$.

Remark 4.2. If the equation (4.31) holds in $(\mathbb{R}^n \setminus \Omega_0)\times(0,T)$, where $\Omega_0 \subset B_R$, and if $w(x,t) = 0$ in $(\mathbb{R}^n \setminus B_R)\times(0,T)$, then $w(x,t) = 0$ in $(\mathbb{R}^n \setminus \Omega_0)\times(0,T)$ by the unique continuation principle (cf. [32]). $\square$

Assume $u_0(x) \in H_2(\mathbb{R}^n \setminus \Omega'(0)),\ u_0(x) = 0$ in $\Omega_0 \setminus \Omega'(0)$, where $\Omega'(0) = \Omega' \cap \{t = 0\}$. There exists a unique solution $u(x,t)$ of (4.26) with the initial data $u(x,0) = u_0(x)$ belonging to the space

$C((0,T),H_2(\mathbb{R}^n\setminus\Omega(t))\cap \overset{\circ}{H}_1(\mathbb{R}^n\setminus\Omega(t))$ (cf., for example, [18]), where $\Omega(t_0)=\Omega'\cap\{t=t_0\}$ and $C((0,T),H_2(\mathbb{R}^n\setminus\Omega(t))\cap \overset{\circ}{H}_1(\mathbb{R}^n\setminus\Omega(t))$ is the space of continuous functions on $[0,T]$ with values in $H_2(\mathbb{R}^n\setminus\Omega(t))\cap \overset{\circ}{H}_1(\mathbb{R}^n\setminus\Omega(t))$, $\overset{\circ}{H}_1(\mathbb{R}^n\setminus\Omega(t))$ consists of functions in $H_1(\mathbb{R}^n\setminus\Omega(t))$ equal to zero on $\partial\Omega(t)$.

Initial-boundary value problem (4.26), (4.27), $u\big|_{\partial\Omega'}=0$, describes an electron confined to the region $\mathbb{R}^n\setminus\Omega(t),\ 0\le t\le T$.

We shall denote, for the brevity, $C((0,T),H_2(\mathbb{R}^n\setminus\Omega(t))\cap \overset{\circ}{H}_1(\mathbb{R}^n\setminus\Omega(t))$, by $W((\mathbb{R}^n\times(0,T))\setminus\Omega')$ and we shall call solutions in $W((\mathbb{R}^n\times(0,T))\setminus\Omega')$ the physically meaningful solutions.

Let $w(x,t)$ be the solution of (4.26) in $(\Omega_0\times(0,T))\setminus\Omega'$ belonging to $C((0,T),H_2(\Omega_0\setminus\Omega(t))\cap \overset{\circ}{H}_1(\Omega_0\setminus\Omega(t))$ where $w(x,0)=0$ in $\Omega_0\setminus\Omega(0)$.

For the brevity, we denote such solutions by $W((\Omega_0\times(0,T))\setminus\Omega')$. It is not clear what is the physical meaning of the solution of (4.26) defined in $(\Omega_0\times(0,T))\setminus\Omega')$ only and having nonzero boundary values on $\partial\Omega_0\times(0,T)$ unless they are the restrictions to $(\Omega_0\times(0,T))\setminus\Omega'$ of the physically meaningful solution from $W((\mathbb{R}^n\times(0,T))\setminus\Omega')$. We shall denote the space of restrictions of $u\in W((\mathbb{R}^n\times(0,T))\setminus\Omega')$ to $((\Omega_0\times(0,T))\setminus\Omega')$ by W_0.

Fortunately, W_0 is dense in $W((\Omega_0\times(0,T))\setminus\Omega')$.

Lemma 4.5 (Density lemma)**.** *Let $w(x,t)\in W((\Omega_0\times(0,T))\setminus\Omega')$, $w(x,0)=0$ in $\Omega_0\setminus\Omega'(0)$. For any ε there exists $u(x,t)\in W((\mathbb{R}^n\times(0,T))\setminus\Omega')$, $u(x,0)=0$ in $\Omega_0\setminus\Omega'(0)$ such that the restriction of $u(x,t)$ to $(\Omega_0\times(0,T))$ satisfies*

$$\sup_{0\le t\le T}[w(x,t)-u(x,t)]_0<\varepsilon,$$

where $[v(x,t)]_0^2=\int\limits_{\Omega_0\setminus\Omega(t)}|v(x,t)|^2dx.$

Proof: Denote by V the Banach space of functions $u(x,t)$ in $(\Omega_0\times(0,T))\setminus\Omega'$ with the norm $\|u\|_V=\int_o^T[u]_0dt$. Let V^* be the dual space with the norm $\|v\|_{V^*}=\sup_{0\le t\le T}[v]_0$. Denote by $\overline{W}_0\subset V^*$ the closure in V^* norm of solutions from $\overline{W}_0$, i.e. the restrictions to $(\Omega_0\times(0,T))\setminus\Omega'$ of functions from $W((\mathbb{R}^n\times(0,T))\setminus\Omega')$.

Let $\overline{W}_0^\perp$ be the set of $v\in V$ such that $(u,v)_0=0$ for all $u\in\overline{W}_0$ where $(u,v)_0$ is the inner product in $L_2((\Omega_0\times(0,T))\setminus\Omega')$. Let f be any element of $\overline{W}_0^\perp$ and f_0 be the extension of f by zero in $(\mathbb{R}^n\setminus\Omega_0)\times(0,T)$.

Denote by $w(x,t)$ the solution of

$$
\begin{aligned}
&-ih\frac{\partial w}{\partial t}+Hw=f_0 \ \text{ in } \ (\mathbb{R}^n\times(0,T))\setminus\Omega',\\
&w(x,T)=0 \ \text{ in } \ \mathbb{R}^n\setminus\Omega'(T),\\
&w\big|_{\partial\Omega'}=0.
\end{aligned}
$$

Note that $w(x,t)\in C((0,T),\overset{\circ}{H}_1(\mathbb{R}^n\setminus\Omega(t)))$ since $f_0\in L_1((0,T),\ L_2(\mathbb{R}^n\setminus\Omega(t)))$. Let (u,w) be L_2-inner product in $(\mathbb{R}^n\times(0,T))\setminus\Omega'$. By the Green's formula in $(\mathbb{R}^n\times(0,T))\setminus\Omega'$ we have

$$
0=(u,f_0)=\big(u,\big(-ih\frac{\partial}{\partial t}+H\big)w\big)=ih\int\limits_{\mathbb{R}^n\setminus\Omega_0}u(x,0)\overline{w}(x,0)dx,
$$

for any $u\in W((\mathbb{R}^n\times(0,T))\setminus\Omega')$, since $-ih\frac{\partial u}{\partial t}+Hu=0,\ u\big|_{\partial\Omega'}=0,\ u(x,0)=0$ for $\Omega_0\setminus\Omega(0)$. Since $u(x,0)\in H_2(\mathbb{R}^n\setminus\Omega')$ is arbitrary on $\mathbb{R}^n\setminus\Omega_0$ we get that

$$
w(x,0)=0 \ \text{ on } \ \mathbb{R}^n\setminus\Omega_0.
$$

Since $w(x,t)$ satisfies $-ih\frac{\partial w}{\partial t}+Hw=0$ in $(\mathbb{R}^n\setminus\Omega_0)\times(0,T)$ and $w(x,0)=w(x,T)=0$ for $x\in\mathbb{R}^n\setminus\Omega_0$, we get, by Lemma 4.4, that $w(x,t)=0$ in $(\mathbb{R}^n\setminus\Omega_0)\times(0,T)$. Therefore the restrictions of $w(x,t))$ and of $\frac{\partial}{\partial\nu}w(x,t)$ to $\partial\Omega_0\times(0,T)$ are equal to zero in the distribution sense (see [20], §24). Let v be any function from $W((\Omega_0\times(0,T))\setminus\Omega')$. Note that $\operatorname{supp}w(x,t)\subset(\overline{\Omega}_0\times[0,T])\setminus\Omega'$.

Hence applying the Green's formula over $(\Omega_0\times(0,T))\setminus\Omega'$ we get

$$
(v,f)_0=\big(v,\big(-ih\frac{\partial}{\partial t}+H\big)w\big)_0=\big(\big(-ih\frac{\partial u}{\partial t}+Hv\big),w\big)_0=0
$$

for any $f\in\overline{W}_0^{\perp}$. Here $(\ )_0$ is the L_2-inner product in $(\Omega_0\times(0,T))\setminus\Omega'$ and we used that $-ih\frac{\partial u}{\partial t}+Hu=0$ and all boundary terms are equal to zero.

Thus $v\in\overline{W}_0$, i.e. for any $\varepsilon>0$ there exists $u(x,t)\in W_0$ such that $\sup_{0\le t\le T}[v-u]_0<\varepsilon$. $\square$

Now we can finish the proof of Theorem 4.3.

Let $u''(x,t)=g_0(x)u'(x,t)$ where $g_0(x)$ is the same as in (4.29). Then $u''(x,t)$ satisfies $-ih\frac{\partial u''}{\partial t}+H''u''=0$ in $(\mathbb{R}^n\times(0,T))\setminus\Omega'$ and $A''(x)=A(x),\ V''(x)=V(x)$ for $x\in(\mathbb{R}^n\setminus\Omega_0)\times(0,T)$ (cf. (4.29)) and $\Lambda''=g_{00}^{-1}\Lambda'g_{00}$ where g_{00} is the restriction of g_0 to $\partial\Omega_0\times(0,T)$.

Let $w=u(x,t)-u''(x,t)=u(x,t)-g_0(x)u'(x,t)$. Then $\big(-ih\frac{\partial w}{\partial t}+Hw\big)=0$ in $(\mathbb{R}^n\setminus\Omega_0)\times(0,T)$ and $w(x,0)=w(x,T)=0$ on $\mathbb{R}^n\setminus\Omega_0$.

Hence, by Lemma 4.4, $w(x,t) = 0$ in $(\mathbb{R}^n \setminus \Omega_0) \times (0,T)$. Therefore,

$$u\big|_{\partial\Omega_0\times(0,T)} = u''\big|_{\partial\Omega_o\times(0,T)} \quad \text{and} \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega_0\times(0,T)} = \frac{\partial u''}{\partial \nu}\Big|_{\partial\Omega_0\times(0,T)} \tag{4.35}$$

for all $u(x,t)$ and $u''(x,t)$ belonging to $W((\mathbb{R}^n \times (0,T)) \setminus \Omega')$. Using the density lemma 4.5 we can extend (4.35) to all u, u'' belonging to $W((\Omega_0 \times (0,T)) \setminus \Omega')$. Therefore $\Lambda = \Lambda''$ on $\partial\Omega_0 \times (0,T)$.

4.4. **Aharonov-Bohm effect for time-dependent electromagnetic potentials.**

When considering AB effect we assume that $B = \operatorname{curl} A = 0$, $E = -\frac{1}{2}\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x} = 0$ in $(\Omega_0 \times (0,T)) \setminus \Omega'$, where B, E are the magnetic and electric fields, Ω' is the union of all obstacles $\Omega'(t) \subset \Omega_0$, $0 \leq t \leq T$. Since $B = E = 0$ we do not need to deal with the complicated X-ray problems and we can substantially relax the restrictions on the obstacles made in Theorem 4.1.

We shall consider the following class of domains in $\mathbb{R}^n \times (0,T)$ that we shall denote by $D^{(1)}$:

Let $0 = T_0 < ... < T_r = T$. Denote by D_{t_0} the intersection of D with the plane $t = t_0$. Then for $t_0 \in (T_{p-1}, T_p)$, $p = 1, ..., r$, we have $D_{t_0} = \Omega_0 \setminus \overline{\Omega}_p(t_0)$, where Ω_0 is a simply-connected domain in $\mathbb{R}^n$, $\Omega_p(t_0) = \bigcup_{j=1}^{m_p} \Omega_{pj}(t_0)$, $\overline{\Omega}_{pj}(t_0) \cap \overline{\Omega}_{pk}(t_0) = \emptyset$ for $j \neq k$, $\overline{\Omega}_{pj}(t_0) \subset \Omega_0$, $\Omega_{pj}(t_0)$ are amooth domains (obstacles). Note that m_p may be different for $p = 1, 2, ..., r$. We assume that $\Omega_p(t_0)$ depends smoothly on $t_0 \in (T_{p-1}T_p)$. We also assume that D_{t_0} depends continuously on $t_0 \in [0,T]$.

Note that some obstacles may merge or split when t_0 crosses T_p, $p = 1, ..., r-1$ (cf. Fig. 4).

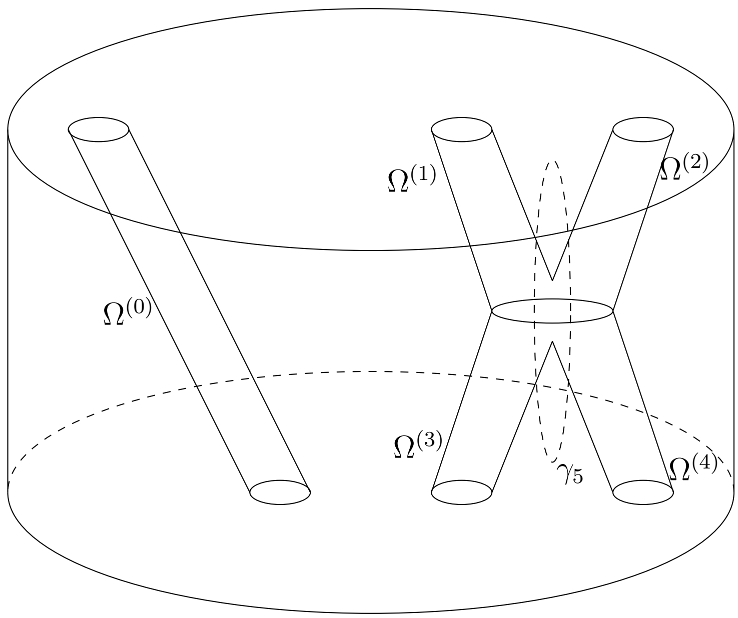

Fig. 4. An example of a domain of class $D^{(1)}$. Obstacles $\Omega^{(3)}$ and $\Omega^{(4)}$ merge, obstacles $\Omega^{(1)}$ and $\Omega^{(2)}$ split.

Note that for each $t_0 \in [0, T]$ the domains $D_{t_0}^{(1)} = D^{(1)} \cap \{t = t_0\}$ are connected. Thus the class of domains $D^{(1)}$ is too restrictive to exhibit the electric AB effect.

We shall introduce a more general class of domains that we call $D^{(2)}$ such that $D_{t_0}^{(2)} = D^{(2)} \cap \{t = t_0\}$ may be not connected on some finite number of intervals in $(0, T)$.

An example of a domain of type $D^{(2)}$ is when we make holes in some obstacles of $D^{(1)}$.

We shall prove first the electromagnetic AB effect in the case of obstacles of the class $D^{(1)}$. Consider the Schrödinger equation (4.1) in $D^{(1)}$, where

$$
\begin{aligned}
&u(x, 0) = 0, \quad x \in D_0^{(1)} = D^{(1)} \cap \{t = 0\}, \\
&u\big|_{\partial\Omega_p(t_0)} = 0, \quad t_0 \in [T_{p-1}, T_p], \ p = 1, ..., r.
\end{aligned}
$$

Let

$$
\alpha = \frac{e}{h} \int\limits_{\gamma} \frac{1}{c} A(x, t) \cdot dx - V(x, t) dt, \tag{4.36}
$$

where γ is a closed curve in $D^{(1)}$. Since we assume that $B = \operatorname{curl} a = 0,\ E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x} = 0$, the integral (4.36), called the electromagnetic flux, depends only on the homotopy class of γ in $D^{(1)}$.

Let $\gamma_1, .., \gamma_l$ be the basis of the homology group of $D^{(1)}$, i.e. any closed curve in $D^{(1)}$ is homotopic to a linear combination of $\gamma_1, ..., \gamma_l$, with integer coefficients. Then the fluxes

$$\alpha_j = \frac{e}{h}\int\limits_{\gamma_j} \frac{1}{c}A(x,t)\cdot dx - V(x,t)dt, \quad j = 1, ..., l,$$

determine the gauge equivalent class of electromagnetic potentials $(A(x,t), V(x,t))$, i.e. $(A(x,t), V(x,t))$ and $(A'(x,t), V'(x,t))$ are gauge equivalent iff $\alpha_j - \alpha_j' = 2\pi m_j,\ m_j \in \mathbb{Z}$, where $\alpha_j' = \frac{e}{h}\int_{\gamma_j}\frac{1}{c}A'(x,t)\cdot dx - V'(x,t)dt$.

As is §4.1. we shall introduce localized geometric optics type solutions $u(x,t)$ of the Schrödinger equation (4.1) in $D^{(1)}$ depending on a large parameter k and satisfying the zero initial condition

$$u(x,0) = 0, \quad x \in D_0^{(1)}, \tag{4.37}$$

and zero boundary conditions on the boundaries of obstacles

$$u(x,t)\big|_{\partial\Omega'} = 0, \tag{4.38}$$

where $\Omega' \subset \mathbb{R}^n \times (0,T)$ is the union of all obstacles $\Omega_p(t),\ t \in [T_{p-1}, T_p], p = 1, ..., r$, and $D_0^{(1)} = \Omega_0 \setminus \Omega_1(0),\ \Omega_1(0) = \Omega' \cap \{t = 0\}$. Such solutions were constructed in [18]. Suppose $t_0 \in (T_{p-1}, T_p),\ 1 \le p \le r$. Suppose $\gamma(x^{(1)}, t_0) = \gamma_0(t_0) \cup ... \cup \gamma_{d-1}(t_0) \cup \gamma_d(x^{(1)}, t_0)$ is a broken ray in D_{t_0} with legs $\gamma_0(t_0), ..., \gamma_{d-1}(t_0), \gamma_d(x^{(1)}, t_0)$, starting at point $x^{(0)} \in \partial\Omega_0$, reflecting at $\partial\Omega_p(t_0)$ and ending at $x^{(1)} \in D_{t_0}^{(1)}$.

As in [18] we can construct an asymptotic solution as $k \to \infty$ of the form (4.15), where supp $u_N(x,t,\omega)$ is contained in a small neighborhood of $x = \gamma(x^{(1)}, t_0), t = t_0$. Note that (cf. [18]) one can find $u^{(N)}(x,t)$ such that $Lu^{(N)} = -Lu_N = O(\frac{1}{k^{N+1}})$ in $D^{(1)}$, $u^{(N)}\big|_{t=0} = 0$, and $u^{(N)}\big|_{\partial\Omega'} = 0,\ u^{(N)}\big|_{\partial\Omega_0\times(0,T)} = 0$ and such that $u^{(N)} = O(\frac{1}{k^{N-2}})$. Here L is the left hand side of (4.1). Then

$$u = u_N + u^{(N)}$$

is the exact solution of $Lu = 0$ in $D^{(1)}$, $u\big|_{t=0} = 0,\ x \in D_0^{(1)}$, $u\big|_{\partial\Omega'} = 0$.

Let $t_0 \in (T_p, T_{p+1})$ and let m_p be the number of the obstacles in $D_{t_0}^{(1)}$. It was proven in [13], [18] that $u(x,t)$ has the following form in

the neighborhood U_0 of $(x^{(1)},t_0)$:

(4.39)
$$u(x,t) = c(x,t)\exp\Big(-i\frac{mk^2t}{2h}+i\frac{mk}{h}\psi_d(x,t)+\frac{ie}{hc}\int_{\gamma(x,t)}A(x,t)\cdot dx\Big) + O\Big(\frac{1}{k}\Big),$$

Here $c(x^{(1)},t_0)\neq 0$ and $\gamma(x,t)$ is a broken ray in $D_t^{(1)}$ that starts at (y,t), (y,t) is close to $(x^{(0)},t_0)$, and such that the first leg of $\gamma(x,t)$ has the same direction as $\gamma_0(t_0)$.

Note the difference between the asymptotic solution (2.45) in §2.6 and the asymptotic solution (4.39). The broken ray $\gamma=\bigcup_{k=0}^d\gamma_k$ in (2.45) is the projection to $\mathbb{R}^2$ of the broken ray $\tilde{\gamma}=\bigcup_{k=1}^d\tilde{\gamma}_k$ in $\mathbb{R}^2\times(0,+\infty)$ having the time variable t as a parameter. The solution (4.39) corresponds to a broken ray $\bigcup_{k=0}^d\gamma_k(t_0)$ in the plane $t=t_0$ with $s=x\cdot\omega_k$ as a parameter on $\gamma_k(t_0)$. $\square$

Let β be the ray $x=x^{(0)}+s\theta,\ s\geq 0,\ t=t_0$, starting at $(x^{(0)},t_0)$ and ending at $(x^{(1)},t_0)$. Choose $x^{(1)}\in\Omega_0$ such that β does not intersect $\Omega'\cap\{t=t_0\}$. We assume that Ω_0 is large enough that such $x^{(1)}$ exists (see Fig.5):

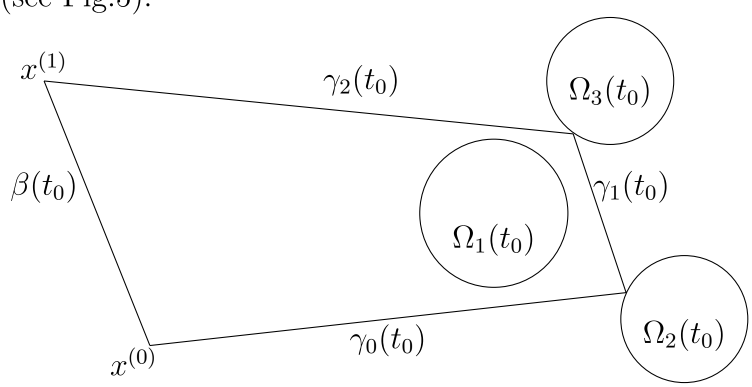

Fig. 5. The broken ray $\gamma=\gamma_0(t_0)\cup\gamma_1(t_0)\cup\gamma_2(t_0)$ and the ray $\beta(t_0)$ belong to $D_{t_0}^{(1)}=D^{(1)}\cap\{t=t_0\}$.

Let $v(x,t)$ be a geometric optics type solution similar to (4.9) and corresponding to the ray β. We have, as in (4.39):
(4.40)
$$v(x,t)=c_1(x,t)\exp\Big(-i\frac{mk^2t}{2h}+i\frac{mk}{h}x\cdot\theta+\frac{ie}{hc}\int_{\beta(x,t)}A(x,t)\cdot dx\Big)+O\Big(\frac{1}{k}\Big).$$

We choose the initial value for $a_0(x,t,\theta)$ (cf. (4.39)) near $(x^{(0)},t_0)$ such that

$$c_1(x^{(1)},t_0) = c(x^{(1)},t_0).$$

Consider $|u(x,t)-v(x,t)|^2$ in a neighborhood $\{(x,t) : |x-x^{(1)}| \le \varepsilon_0,\ |t-t_0|<\varepsilon_0\}$.

As in §2.6 we get for a small neighborhood of $(x^{(1)},t_0)$

$$|u(x,t,\omega)-v(x,t,\theta)|^2 = |c(x^{(1)},t_0)|^2\ 4\sin^2\frac{\alpha(t_0)}{2} + O(\varepsilon), \tag{4.41}$$

where

$$\alpha(t_0) = \frac{e}{hc}\Big(\int_{\gamma(x^{(1)},t_0)} A(x,t_0)\cdot dx - \int_{\beta(x^{(1)},t_0)} A(x,t_0)\cdot dx\Big).$$

Note that $\alpha(t_0)$ is the sum of the fluxes of those obstacles $\Omega_{pj}(t_0),\ 1\le j\le m_p$, that are encircled by $\gamma\cup\beta$. As in §2.6, varying γ and β at least m_p times we can recover $\alpha_{pj}(t_0)$ (modulo $2\pi n$), $1\le j\le m_p$, up to a sign, where

$$\alpha_{pj}(t_0) = \frac{e}{hc}\int_{\gamma_{pj}(t_0)} A\cdot dx, \quad 1\le j\le m_p, \tag{4.42}$$

and $\gamma_{pj}(t_0)$ is a simple contour in $D_{t_0}^{(1)}$ encircling $\Omega_{pj}(t_0)$, $1\le j\le m_p$. Note that α_{pj} are the same for any $t_0\in(T_p,T_{p+1})$. We can repeat the same arguments for any $t_0\ne T_1,...,T_{p-1}$.

Our class of time-dependent obstacles is such that $D_{t_0}^{(1)}$ is connected for any $t_0\in[0,T]$. It follows from this assumption that a basis of the homology group of $D^{(1)}$ is contained in the set $\gamma_{pj}(t_p),\ 1\le j\le m_p,\ t_p\in (T_{p-1},T_p),\ 1\le p\le r$, of "flat" closed curves that are contained in the planes t=const.

Denote such basis by $\gamma^{(1)}(t^{(1)}),...,\gamma^{(l)}(t^{(l)})$. Then any closed contour γ in D is homotopic to a linear combination $\sum_{j=1}^l n_j\gamma^{(j)}(t^{(j)})$ where $n_j\in\mathbb{Z}$. Therefore the flux

$$\frac{e}{h}\int_\gamma \frac{1}{c}A\cdot dx - V\,dt = \sum_{j=1}^l n_j\alpha^{(j)}(t^{(j)}), \tag{4.43}$$

where $\alpha^{(j)}(t^{(j)}) = \frac{e}{hc}\int_{\gamma^{(j)}(t^{(j)})} A\cdot dx$.

Thus the fluxes $\alpha^{(j)}(t^{(j)}),\ 1\le j\le l$, mod $2\pi n,\ n\in\mathbb{Z}$, determine the gauge equivalence class of $(A(x,t),V(x,t))$. Therefore computing the probability densities of appropriate solutions we are able to determine the gauge equivalence classes of electromagnetic potentials up to a sign.

The solution $u(x,t,\omega)-v(x,t,\theta)$ in (4.41) is a solution of the Schrödinger equation in $D^{(1)}$ with nonzero boundary conditions on $\partial\Omega_0\times(0,T)$. The

probability density $|u(x,t,\omega)-v(x,t,\theta)|^2$ depends on the flux $\alpha(t_0)$ but this does not prove yet that the magnetic flux makes the physical impact since $u(x,t,\omega)-v(x,t,\theta)$ may be not a physically meaningful solution. However, by the density lemma 4.5 there exists a physically meaningful solution $v_{\varepsilon_1} \in W_0$ such that

$$\max_{0\leq t\leq T} \int\limits_{D_{t_0}^{(1)}} |u-v-v_{\varepsilon_1}|^2 dx < \varepsilon_1,$$

where ε_1 is much smaller than $\varepsilon > 0$ in (4.41). Then

$$\int\limits_{U_0} |v_\varepsilon(x,t_0)|^2 = |c^{(1)}(x^{(1)},t_0)|^2 4\sin^2\frac{\alpha(t_0)}{2}\mu(U_0) + O(\varepsilon),$$

where $\mu(U_0)$ is the volume of U_0, i.e. $\int_{U_0} |v_\varepsilon(x,t_0)|^2 dx$ depends on $\alpha(t_0)$. Thus we proved AB effect since $v_\varepsilon(x,t)$ is a physically meaningful solution.

Example 4.1. Consider the domain shown in Fig.4. Let γ_p, $0 \leq p \leq 4$, be simple closed curves encircling $\Omega^{(p)}$. There is also a simple closed curve γ_5 that is not homotopic to any closed curve contained in the plane $t = \text{const}$. Note that $\gamma_1+\gamma_2 \approx \gamma_3+\gamma_4$ where $\approx$ means homotopic. Also $\gamma_5 \approx \gamma_1-\gamma_3$. Therefore $\gamma_0,\gamma_1,\gamma_2,\gamma_3$ is a basis of the homology group of $D^{(1)}$.

Let α_j be the fluxes corresponding to γ_j. Note that if γ_j is flat then $\alpha_j = \int_{\gamma_j} A\cdot dx$ is a magnetic flux. However $\alpha_5 = \frac{e}{\hbar}\int_{\gamma_5}\frac{1}{c}A\cdot dx - V dt$ is an electromagnetic flux. Since $\gamma_5 \approx \gamma_1-\gamma_3$ we have that $\alpha_5 = (\alpha_1-\alpha_3) \pmod{2\pi n}$, $n\in\mathbb{Z}$. $\square$

4.5. **Combined electric and magnetic AB effect.**

In this subsection we consider domains of the class $D^{(2)}$ that will allow to study the combined electric and magnetic AB effect (cf. Markovitch et al [38] and [22]).

Example 4.2. We shall start with the example of the domain $D = (\mathbb{R}^n\times(0,T))\setminus\Omega_0$ of class $D^{(2)}$ that was considered in §3 (see Fig.3). In §3 we assumed that $A(x)=0$. Now we shall consider the Schrödinger equation of the form (4.26) with $A(x,t), V(x,t)$ such that $B = \operatorname{curl} A = 0$, $E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x} = 0$ in D.

Denote by Q the cylinder $Q = \{R_1 \leq |x| \leq R_2,\ \varepsilon \leq t \leq T-\varepsilon\}$. Note that $D_{t_0} = D\cap\{t=t_0\}$ is connected for $0\leq t_0<\varepsilon$ and $T-\varepsilon<t_0\leq T$ and has two connected components when $\varepsilon\leq t_0\leq T-\varepsilon$, one of them being $Q = \{R_1\leq|x|\leq R_2\}$.

Since Q is simply-connected and $\operatorname{curl} A = 0$ in Q, we can find $\varphi(x,t) \in C^\infty(Q)$ such that $A(x,t) = \frac{\partial \varphi(x,t)}{\partial x}$ in Q.

Making the gauge transformation with the gauge $e^{\frac{ie}{hc}\varphi(x,t)}$, we can replace (4.26) with a gauge equivalent equation such that $\hat{A}(x,t) = 0$ in Q and $\hat{V}(x,t) = V(x,t) + \frac{1}{c}\frac{\partial \varphi}{\partial t}$. Thus $\frac{\partial \hat{V}(x,t)}{\partial x} = 0$ in Q since $E = 0$, and we get that $\hat{V}(x,t) = \hat{V}_0(t)$ in Q.

Therefore, without loss of generality we can, from the beginning, assume $V(x,t) = V_0(t)$ in Q, $A = 0$ in Q.

The basis of the homology group for D consists of $\gamma_1 = \{|x| = R_2+1\}$ and of a closed curve δ_1 in the (x_1, t) plane that encircles the rectangle $\{R_1 \leq x_1 \leq R_2, \varepsilon \leq t \leq T-\varepsilon, x_2 = 0\}$.

Potentials $(A,V), (A',V')$ are gauge equivalent if

$$\alpha_1 - \alpha_1' = 2\pi n_1,\ n_1 \in \mathbb{Z},\quad \alpha_2 - \alpha_2' = 2\pi n_2,\ n_2 \in \mathbb{Z},$$

where $\alpha_1 = \int_{\gamma_1} A(x)\cdot dx$, $\alpha_1' = \int_{\gamma_1} A'(x)\cdot dx$, $\alpha_2 = \int_{\delta_1} A(x)\cdot dx - Vdt$, $\alpha_2' = \int_{\delta_1} A'(x)\cdot dx - V'dt$.

We shall prove that $(A,V), (A',V')$ made a different physical impact if either $\alpha_1 - \alpha_1' \neq 2\pi n$, $\forall n \in \mathbb{Z}$, and $\alpha_1 + \alpha_1' \neq 2\pi n$, $\forall n$, or if $\alpha_1 - \alpha_1' = 2\pi n_1$, and $\alpha_2 - \alpha_2' \neq 2\pi n$, $\forall n \in \mathbb{Z}$. This will prove the combined AB effect.

It follows from the results of §4.4 that $(A,V), (A',V')$ have a different physical impact if $\alpha_1 - \alpha_1' \neq 2\pi n$, $\forall n \in \mathbb{Z}$, and $\alpha_1 + \alpha_1' \neq 2\pi n$, $\forall n \in \mathbb{Z}$.

Suppose $\alpha_1 - \alpha_1' = 2\pi n_1,\ n_1 \in \mathbb{Z}$

Then for each $t_0 \in [0,T]$ there exists $g(x,t_0)$ such that

$$\frac{e}{c}A(x,t) = \frac{e}{c}A'(x,t) - ih\frac{\partial g}{\partial x}g^{-1}.$$

Using the gauge $g(x,t)$ we transform $\big(ih\frac{\partial}{\partial t} - H'\big)u' = 0$ to $\big(ih\frac{\partial}{\partial t} - H''\big)u'' = 0$, where $A''(x,t) = A(x,t)$, i.e. H and H'' have the same magnetic potentials in $D \setminus Q$. Since $E = -\frac{1}{c}\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x} = 0$ and $E'' = -\frac{1}{c}\frac{\partial A''}{\partial t} - \frac{\partial V''}{\partial x} = 0$ we get that $\frac{\partial V}{\partial x} - \frac{\partial V''}{\partial x} = 0$. Hence $V - V'' = 0$ in $D \setminus Q$ since $V = V'' = 0$ for large $|x|$. Thus $A = A'', V = V''$ outside of Q, $A = A'' = 0$ in Q, $V = V_0(t)$, $V'' \equiv V_0''(t)$ in Q.

Note that $\alpha_2 - \alpha_2' = \frac{e}{h}\int_\varepsilon^{T-\varepsilon}(V_0(t) - V_0''(t))dt$ since $H = H''$ outside of Q. If $\frac{e}{h}\int_\varepsilon^{T-\varepsilon} V_0(t)dt - \frac{e}{h}\int_\varepsilon^{T-\varepsilon} V_0''(t)dt \neq 2\pi n$, $\forall n \in R$, we shall prove that $V_0(t)$, $V_0''(t)$ have different physical impacts.

We shall use the same arguments as in §3. The difference is that $A \neq 0$, $V \neq 0$ outside of Q for u and u''. We have $u(x, T-\varepsilon) = u''(x, T-\varepsilon)$ for $|x| > R_2$ since $u(x,t)$ and $u''(x,t)$ satisfy the same equation, and the initial and boundary conditions for u and u'' are the same when $t < T-\varepsilon, |x| > R_2$. Suppose $|u(x,t)| = |u'(x,t)| = R$

in U_0, where U_0 is the same as in §3. Using the polar representation $u = Re^{i\Phi}, u'' = Re^{i\Phi''(x,t)}$, separating the real part, we get

$$h\Phi_t = \frac{h^2}{2m}(\Delta R - R|\nabla\Phi|^2) + \frac{eh}{mc}A \cdot \nabla\Phi R + \Big(\frac{e^2}{2mc^2}A^2 + eV\Big)R$$

in U_0. The equation for Φ'' is the same since $R'' = R$. The initial condition in $U_0 \cap \{t = T - \varepsilon\}$ for Φ and Φ'' are also the same since $u(x, T - \varepsilon) = u''(x, T - \varepsilon)$. Therefore $\Phi = \Phi''$ in U_0. The continuation of the proof is the same as in §3. $\square$

Consider now the equation (4.26) in a general domain of the form $D^{(2)}, B = E = 0$ in $D^{(2)}$. Let $Q_j, j = 1, 2, ..., d$, be such that $Q_{jt_0} = Q_j \cap \{t = t_0\}$ is a bounded connected component of $D^{(2)}_{t_0} = D^{(2)} \cap \{t = t_0\}$ for $\varepsilon_j \le t_0 \le T'_j,\ 0 < T'_1 < T'_2 < ... < T'_d < T,\ j = 1, ..., d$.

As in the previous example, we may assume that $A = 0$ in $Q_j,\ V = V_j(t)$ in $Q_j,\ 1 \le j \le d$.

The basis of the homology group $D^{(2)}$ consists of the basis $\gamma_1, ..., \gamma_l$ of the connected domain $D^{(2)} \setminus \bigcup_{j=1}^d Q_j$ and curves $\delta_1, ..., \delta_d$ similar to δ_1 in Example 4.2, passing through the holes $Q_1, ..., Q_d$.

Let $\alpha_j = \frac{e}{h}\int_{\gamma_j} \frac{1}{c}A \cdot dx - V dt,\ \beta_k = \frac{e}{h}\int_{\delta_k} \frac{1}{c}A \cdot dx - V dt$ be the electromagnetic fluxes.

We shall show that (A, V) and (A', V') have a different physical impact if

a) either $\alpha_j - \alpha'_j \ne 2\pi n,\ \forall n \in \mathbb{Z}$ and $\alpha_j + \alpha'_j \ne 2\pi n,\ \forall n \in Z$, for some $j,\ 1 \le j \le l$,

or

b) $\alpha_j - \alpha'_j = 2\pi n_j,\ j = 1, ..., l$ and $\beta_k - \beta'_k \ne 2\pi n_k,\ \forall n_k \in \mathbb{Z}$, for some $k,\ 1 \le k \le d$.

Here α'_j, β'_k are fluxes for (A', V').

Assertion a) follows from the results of §4.4.

If $\alpha_j - \alpha'_j = 2\pi n_j,\ 1 \le j \le l$, we can, as in Example 4.2, replace (A', V') by a gauge equivalent (A'', V'') such that $A = A'', V = V''$ in $D^{(2)} \setminus \bigcup_{j=1}^d Q_j$.

If $\beta_1 - \beta'_1 \ne 2\pi n, \forall n \in \mathbb{Z}$, then we obtain, as in the proof of Example 4.2, that $|u(x,t)|^2 \ne |u''(x,t)|^2$ for $t > T'_1$ and thus we prove the AB effect.

If $\beta_1 - \beta'_1 = 2\pi n, n \in \mathbb{Z}$, but $\beta_2 - \beta'_2 \ne 2\pi n,\ \forall n \in \mathbb{Z}$, we get that $u(x,t) = u''(x,t)$ for $T'_1 < t < T'_2$, but $|u(x,t)| \ne |u''(x,t)|$ for $t > T'_2$, etc. Thus AB effect holds if $\beta_j - \beta'_j \ne 2\pi n,\ \forall n$, for one of $1 \le j \le d$.

5. Gravitational AB effect

In this section we shall study the gravitational analog of the quantum mechanical AB effect.

5.1. **Global isometry.**

Consider a pseudo-Riemannian metric $\sum_{j,k=0}^{n} g_{jk}(x)dx_{jk}dx_k$ with Lorentz signature in Ω, where $x_0 \in \mathbb{R}$ is the time variable, $x = (x_1, ..., x_n) \in \Omega$, $\Omega = \Omega_0 \setminus \cup_{j=1}^{m} \overline{\Omega}_j$, Ω_0 is simply connected, $\overline{\Omega}_j \subset \Omega_0$, Ω_j, $1 \leq j \leq m$, are obstacles (cf. subsection 3.3). We assume that $g_{jk}(x)$ are independent of x_0, i.e. the metric is stationary.

Consider a group of transformations (changes of variables)

$$x' = \varphi(x), \qquad x_0' = x_0 + a(x), \tag{5.1}$$

where $x' = \varphi(x)$ is a diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega'} = \varphi(\overline{\Omega})$ and $a(x) \in C^\infty(\overline{\Omega})$. Two metrics $\sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k$ and $\sum_{j,k=0}^{n} g'_{jk}(x')dx'_j dx'_k$ are called isometric if

$$\sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k = \sum_{j,k=0}^{n} g'_{jk}(x')dx'_j dx'_k, \tag{5.2}$$

where (x_0', x') and (x_0, x) are related by (5.1).

The group of isomorphisms (isometries) will play the same role as the gauge group for the magnetic AB effect.

Let

$$\square_g u(x_0, x) = 0 \ \text{ in } \ \mathbb{R} \times \Omega$$

be the wave equation corresponding to the metric g, i.e.

$$\square_g u \overset{def}{=} \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g_0}} \frac{\partial}{\partial x_j}\Big(\sqrt{(-1)^n} g_0 g^{jk}(x) \frac{\partial u}{\partial x_k}\Big) = 0, \tag{5.3}$$

where $g_0 = \det[g_{jk}]_{j,k=0}^{n}$, $[g^{jk}(x)] = [g_{jk}]^{-1}$.

Solutions of (5.3) are called gravitational waves on the background of the space-time with the metric g.

Consider the initial boundary value problem for (5.3) in $\mathbb{R} \times \Omega$ with zero initial conditions

$$u(x_0, x) = 0 \ \text{ for } x_0 \ll 0, \ x \in \Omega, \tag{5.4}$$

and the boundary condition

$$u\big|_{\mathbb{R}\times\partial\Omega_0} = f, \quad u\big|_{\mathbb{R}\times\partial\Omega_j} = 0, \quad 1 \leq j \leq m, \tag{5.5}$$

where $f \in C_0^\infty(\mathbb{R} \times \partial\Omega_0)$. Let Λ_g be the Dirichlet-to-Neumann (DN) operator, i.e. $\Lambda_g f = \frac{\partial u}{\partial \nu_g}\Big|_{\mathbb{R}\times\partial\Omega_0}$, where

$$\frac{\partial u}{\partial \nu_g} = \sum_{j,k=0}^{n} g^{jk}(x)\nu_j(x)\frac{\partial u}{\partial x_k}\Big(\sum_{p,r=0}^{n} g^{pr}(x)\nu_p\nu_r\Big)^{-\frac{1}{2}}. \tag{5.6}$$

Here $u(x_0, x)$ is the solution of (5.3), (5.4), (5.5), $\nu(x) = (\nu_1, ..., \nu_n)$ is the outward unit normal to $\partial\Omega_0$, $\nu_0 = 0$.

Let Γ be an open subset of $\partial\Omega_0$. We shall say that boundary measurements are taken on $(0, T) \times \Gamma$ if we know the restriction $\Lambda_g f\big|_{(0,T)\times\Gamma}$ for any $f \in C_0^\infty((0, T) \times \Gamma)$.

Consider metric g' in Ω' and the corresponding initial-boundary value problem

$$\Box_{g'} u'(x_0', x') = 0 \ \text{ in } \ \mathbb{R} \times \Omega', \tag{5.7}$$

$$u'(x_0', x') = 0 \ \text{ for } \ x_0' \ll 0, \ x' \in \Omega', \tag{5.8}$$

$$u\big|_{\mathbb{R}\times\partial\Omega_0'} = f, \quad u'\big|_{\mathbb{R}\times\partial\Omega_j'} = 0, \ 1 \leq j \leq m', \tag{5.9}$$

where $\Omega' = \Omega_0' \setminus \bigcup_{j=1}^{m'} \overline{\Omega_j'}$.

We assume that $\partial\Omega_0 \cap \partial\Omega_0' \neq \emptyset$. Let Γ be an open subset of $\partial\Omega_0 \cap \partial\Omega_0'$.

The following theorem was proven in [19] (see [19], Theorem 2.3).

Theorem 5.1. *Suppose $g^{00}(x) > 0$, $g_{00}(x) > 0$ in $\overline{\Omega}$ and $(g')^{00} > 0$, $g_{00}' > 0$ in $\overline{\Omega'}$. Suppose $\Lambda_g f\big|_{(0,T)\times\Gamma} = \Lambda_{g'} f\big|_{(0,T)\times\Gamma}$ for all $f \in C_0^\infty((0, T) \times \Gamma)$. Suppose $T > T_0$, where T_0 is sufficiently large. Then metrics g and g' are isometric, i.e. there exists a change of variables (5.1) such that (5.2) holds. Moreover, $\varphi\big|_\Gamma = I$, $a\big|_\Gamma = 0$.*

If two metrics g and g' in Ω and Ω', respectively, are isometric, then the solutions $u(x_0, x)$ and $u(x_0', x')$ of the corresponding wave equations are the same after the change of variables (5.1). Therefore isometric metrics have the same physical impact.

Suppose two metric g and g' are isometric in some neighborhood $V \subset \Omega, \overline{V} \cap \partial\Omega \neq \emptyset$. Let $\Gamma \subset \overline{V} \cap \partial\Omega$. There exists a local isomorphism

$$x' = \varphi_V(x), \quad x_0' = x_0 + a_V(x) \tag{5.10}$$

that transforms g' to the metric $\hat{g}$ in $\overline{V}$ such that $\hat{g} = g$ in V. Extend the isometry (5.10) from $\overline{V}$ to $\overline{\Omega}$ and denote by $\hat{g}$ the image of g' under this map. Thus $\hat{g}$ isometric to g' in Ω and $\hat{g} = g$ in $\overline{V}$.

Theorem 5.2. *The metrics g and $\hat{g}$ are not isometric if and only if the boundary measurements*

$$\Lambda_g f\big|_{(0,T)\times\Gamma} \neq \Lambda_{\hat{g}} f\big|_{(0,T)\times\Gamma} \quad \textit{for some} \quad f \in C_0^\infty((0, T) \times \Gamma), \tag{5.11}$$

Proof (cf. §2.1): Suppose g and $\hat{g}$ are not isometric. If $\Lambda_g f\Big|_{(0,T)\times\Gamma} = \Lambda_{\hat{g}} f\Big|_{(0,T)\times\Gamma}$ for all $f \in C_0^\infty((0,T)\times\Gamma)$ then by Theorem 5.1 there exists a map of the form (5.1) that transforms $\hat{g}$ to g and such that

$$\varphi\Big|_\Gamma = I, \quad a\Big|_\Gamma = 0. \tag{5.12}$$

Since $g = \hat{g}$ in $\overline{V}$ any such map satisfies (5.12). Thus g and $\hat{g}$ are isometric, i.e. we got a contradiction. Therefore if g and $\hat{g}$ are not isometric then (5.11) holds.

Vice versa, suppose g and $\hat{g}$ are isometric, i.e. (5.1) holds. Then for all solutions $u(x_0,x)$ and $\hat{u}(\hat{x}_0,\hat{x})$ of equations (5.3), (5.4), (5.5) and (5.7), (5.8), (5.9), respectively, we have $u(x_0,x) = \hat{u}(\hat{x}_0,\hat{x})$, where (x_0,x) and $(\hat{x}_0',\hat{x})$ are related by (5.1). Note that (5.12) also holds since $g = \hat{g}$ in $\overline{V}$. Thus we have $\Lambda_g f\big|_{(0,T)\times\Gamma} = \Lambda_{\hat{g}} f\big|_{(0,T)\times\Gamma}$ for all $f \in C_0^\infty((0,T)\times\Gamma)$. Therefore if (5.11) holds then g and $\hat{g}$ are not isometric.

It follows from (5.11) that non-isometric metrics g and $\hat{g}$ (and therefore g and g') have different physical impacts.

Note that the open set Γ can be arbitrary small. However the time interval $(0,T)$ must be large enough: $T > T_0$.

5.2. **Locally static stationary metrics.** Let g and g' be isometric. Substituting $dx_0' = dx_0 + \sum_{j=1}^n a_{x_j}(x)dx_j$ and taking into account that dx_0 is arbitrary, we get from (5.1) and (5.2) that

$$g_{00}'(x') = g_{00}(x), \tag{5.13}$$

$$2g_{00}'(x')\sum_{j=1}^n a_{x_j}(x)dx_j + 2\sum_{j=1}^n g_{j0}'(x')dx' = 2\sum_{j=1}^n g_{j0}(x)dx_j. \tag{5.14}$$

Using (5.13) we can rewrite (5.14) in the form

$$\sum_{j=1}^n \frac{1}{g_{00}'(x')} g_{j0}'(x')dx' = \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j - \sum_{j=1}^n a_{x_j}(x)dx_j. \tag{5.15}$$

Let γ be an arbitrary closed curve in Ω, and let γ' be the image of γ in Ω' under the map (5.1). Integrating (5.15) we get

$$\int_{\gamma'} \sum_{j=1}^n \frac{1}{g_{00}'(x')} g_{j0}'(x')dx' = \int_\gamma \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j, \tag{5.16}$$

since $\int_\gamma \sum_{j=1}^n a_{x_j}(x)dx_j = 0$. Therefore the integral

$$\alpha = \int_\gamma \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j \tag{5.17}$$

is the same for all isometric metrics. $\square$

A stationary metric g is called static in Ω if it has the form

$$g_{00}(x)(dx_0)^2 + \sum_{j,k=1}^n g_{jk}(x)dx_j dx_k, \tag{5.18}$$

i.e. when $g_{0j}(x) = g_{j0}(x) = 0,\ 1 \le j \le n,\ x \in \Omega$.

Suppose the stationary metric $g(x)$ in Ω is locally static, i.e. for any point in Ω there is a neighborhood V such that the isometry $x_0' = x_0 + a_V(x), x' = x$ transforms the metric g restricted to V to some static metric $g_{00}'(x)(dx_0')^2 + \sum_{j,k=1}^n g_{jk}'(x)dx_j dx_k$, i.e. $g_{j0}'(x) = g_{jk}(x) - a_{Vx_j}(x) = 0, 1 \le j \le n, x \in V$.

Suppose that metric g is not globally static in Ω, i.e. there is no $a(x) \in C^\infty(\overline{\Omega})$ such that $x_0' = x_0 + a(x), x' = x$, transforms g to a static metric g' globally in Ω, i.e. g and g' are not isometric. Then Theorem 5.2 implies that $\Lambda_g f\big|_{\Gamma\times(0,T)} \ne \Lambda_{g'} f\big|_{\Gamma\times(0,T)}$ for some $f \in C_0^\infty(\Gamma)$, i.e. metric g and g' have a different physical impact. This proves the gravitational AB effect.

Note that $\int_\gamma \sum_{j=1}^n \frac{1}{g_{00}(x)} g_{j0}(x)dx_j = 0$ for any $\gamma \subset V$ if g is locally isometric to a static metric in V. If g is not globally isometric to a static metric then integral (5.17) may be not zero. It plays a role of the magnetic flux for the magnetic AB effect and α in (5.17) depends only on the homotopic class of γ when g is locally static.

This formulation of the gravitational AB effect was given by Stachel in [46] who proved it for some explicit class of locally static but globally not static metrics.

5.3. **A new inverse problem for the wave equation.** Let g and g' be two stationary metrics in $\mathbb{R}^n \setminus \cup_{j=1}^m \Omega_j$ such that

$$g_{jk}(x) = g_{jk}'(x) \quad \text{for} \quad |x| > R, \tag{5.19}$$

where R is large. Assume also that

$$g_{jk}(x) = \eta_{jk} + h_{jk}(x) \quad \text{for} \quad |x| > R, \tag{5.20}$$

where

$$h_{jk}(x) = O\Big(\frac{1}{|x|^{1+\varepsilon}}\Big) \quad \text{for} \quad |x| > R, \varepsilon > 0,$$

$$\sum_{j,k=1}^{n} \eta_{jk} dx_j dx_k = dx_0^2 - \sum_{j=1}^{n} dx_j^2$$

is the Minkowski metric and $h_{jk}(x) = O(\frac{1}{|x|^{1+\varepsilon}})$, $\varepsilon > 0$, for $|x| > R$.

The following theorem is analogous to Theorem 4.3.

Theorem 5.3. *Let $\square_g u = 0$ and $\square_{g'} u' = 0$ in $(0,T) \times (\mathbb{R}^n \setminus \bigcup_{j=1}^m \Omega_j)$, where $T > T_0$ (cf. Theorem 5.1). Suppose (5.19) and (5.20) hold. Consider two initial-boundary value problems with the same initial conditions*

$$u(0,x) = u_0(x), \qquad u'(0,x) = u_0(x),$$

$$u_t(0,x) = u_1(x), \quad u_t'(0,x) = u_1(x), \quad x \in \mathbb{R}^n \setminus \bigcup_{j=1}^{n} \overline{\Omega_j},$$

$$u\big|_{(0,T)\times\partial\Omega_j} = 0, \quad u'\big|_{(0,T)\times\partial\Omega_j} = 0, \quad 1 \le j \le m,$$

$$u_0(x) = u_1(x) = 0 \quad \text{in } B_R \setminus \bigcup_{j=1}^{m} \Omega_j,$$

where $B_R = \{x : |x| < R\}$. Suppose $g_{00}(x) > 0$, $g_{00}'(x) > 0$, $g^{00}(x) > 0$, $(g')^{00} > 0$ in $\mathbb{R}^n \setminus \bigcup_{j=1}^n \Omega_j$. If $u_0(x) \in \overset{\circ}{H}_1(\mathbb{R}^n \setminus \bigcup_{j=1}^m \Omega_j))$, $u_1(x) \in L_2(\mathbb{R}^n \setminus \bigcup_{j=1}^m \Omega_j)$ and if

$$u(T,x) = u'(T,x), \quad u_{x_0}(T,x) = u_{x_0}'(T,x), \quad x \in \mathbb{R}^n \setminus B_R,$$

for all $u_0(x)$ and $u_1(x)$, then metrics g and g' are isometric in $\mathbb{R}^n \setminus \cup_{j=1}^n \Omega_j$.

Proof: It follows from the existence and uniqueness theorem that the solutions $u(x_0,x)$ and $u'(x_0,x)$ belong to $H_1((0,T)\times(\mathbb{R}^n \setminus \bigcup_{j=1}^n \Omega_j))$. Let $v = u(x_0,x) - u'(x_0,x)$. Then $\square_g v = 0$ in $(0,T) \times (\mathbb{R}^n \setminus B_R)$ and $v(0,x) = v_{x_0}(0,x) = 0$, $v(T,x) = v_{x_0}(T,x) = 0$ for $x \in \mathbb{R}^n \setminus B_R$. Extend $v(x_0,x)$ by zero for $x_0 > T$ and $x_0 < 0$ and make the Fourier transform in $x_0 : \tilde{v}(\xi_0,x) = \int_{-\infty}^{\infty} v(x_0,x) e^{-ix_0\xi_0} dx_0$. Then $\tilde{v}(\xi_0,x)$ belongs to $L_2(\mathbb{R}^n \setminus B_R)$ for all $\xi_0 \in \mathbb{R}$ and satisfies the equation

$$L\big(i\xi_0, \frac{\partial}{\partial x}\big)\tilde{v}(\xi_0,x) = 0, \quad x \in \mathbb{R}^n \setminus B_R,$$

where $L\big(i\xi_0, i\xi\big)$ is the symbol of $\square_g$.

It follows from Hörmander ([31]) that $\tilde{v}(\xi_0, x) = 0$ in $\mathbb{R}^n \setminus B_R$ for all ξ_0. Therefore $u(x_0, x) = u'(x_0, x)$ for $x_0 \in (0,T),\ x \in \mathbb{R}^n \setminus B_R$. Then $u\big|_{(0,T)\times\partial B_R} = u'\big|_{(0,T)\times\partial B_R} \in H_{\frac{1}{2}}((0,T)\times\partial B_R)$ and $\frac{\partial u}{\partial\nu_g}\big|_{(0,T)\times\partial B_R} = \frac{\partial u'}{\partial\nu_g}\big|_{(0,T)\times\partial B_R} \in H_{-\frac{1}{2}}((0,T)\times\partial B_R)$ (cf. [20], §23), i.e. the boundary measurements of u and u' on $(0,T)\times\partial B_R$ are the same.

Analogously to the proof of Lemma 4.5 one can show that $u\big|_{(0,T)\times\partial B_R}$ and $u'\big|_{(0,T)\times\partial B_R}$ are dense in $H_{-\frac{1}{2}}((0,T)\times\partial B_R)$. Hence the DN operators Λ and Λ' are equal on $(0,T)\times\partial B_R$. Thus Theorem 5.1 implies that g and g' are isometric.

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